Weighted Haar and Alpert Wavelets: Degeneracy and Stability

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Haar Wavelets

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- For $I \in D$ define $h^I = \frac{1}{\sqrt{|I|}}(\chi_{I_{\text{left}}} - \chi_{I_{\text{right}}})$.
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Properties:

- $\{h^I\}_{I\in D}$ is an orthonormal basis for $L^2(\mathbb{R})$.
- Telescoping: for any $f \in L^2(\mathbb{R})$

$$\sum_{2^{m+1}\leq|I|\leq 2^n} \langle f, h^I\rangle h^I = \sum_{|I|=2^m} \langle f, \frac{1}{2^m}\chi_I\rangle\chi_I - \sum_{|I|=2^n} \langle f, \frac{1}{2^n}\chi_I\rangle\chi_I.$$

- Moment vanishing: each h^I has $\int_{\mathbb{R}} h^I\, dx = 0$.
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Weighted Haar Wavelets

Let μ be a locally finite positive Borel measure.

- For each $I \in D$ define

$$h^I = \frac{1}{\sqrt{\mu(I)}}\left(\sqrt{\frac{\mu(I_{\text{right}})}{\mu(I_{\text{left}})}}\chi_{I_{\text{left}}} - \sqrt{\frac{\mu(I_{\text{left}})}{\mu(I_{\text{right}})}}\chi_{I_{\text{right}}}\right).$$

- Weighted Haar bases retain orthonormality, telescoping, and moment vanishing.
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Alpert Wavelets

Let D be a dyadic grid on $\mathbb{R}$ and $k \geq 1$ be an integer.

- For each $I \in D$ choose k functions which are:
  - polynomial of degree less than k on children of I.
  - orthogonal to polynomials of degree less than k.
- The union over all $I \in D$ of such functions forms an Alpert basis for $L^2(\mathbb{R})$.
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Tops of Dyadic Grids

Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$.

- Let D be a dyadic grid and consider an increasing nested tower of dyadic cubes $Q_1 \subset Q_2 \subset Q_3 \subset \dots$.
- A top T of a dyadic grid is the limit of such a tower.
- A dyadic grid has at least 1 top and at most 2^n tops, and is partitioned by its tops.
- Each top is an "infinite cube" which grows in either one or both directions parallel to each coordinate axis.
- If any monomial of degree less than k is square-integrable on an entire top, it must be included in the weighted Alpert basis (Alexis, Sawyer, and Uriarte-Tuero, 2022).

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Let $Q \in D$ be a dyadic cube in $\mathbb{R}^n$ and $k \geq 1$ be an integer.

- All monomials in Lebesgue measure are linearly independent.
- There are $\binom{n+k-1}{k-1}$ monomials in n variables of degree less than k
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Example

Let $k = 1$, D be the standard dyadic grid on $\mathbb{R}$, and μ be the measure with point masses at $\frac{1}{4}$ and $\frac{3}{4}$ and no mass elsewhere.

- For the dyadic interval $[0, 1)$ we have $\dim L^2_{[0,1),1}(\mu) = 1$.
- For any dyadic interval $I \neq [0, 1)$, $\dim L^2_{I,1}(\mu) = 0$.
- $L^2(\mu)$ has dimension 2 and is spanned by a constant function and a Haar wavelet on $[0, 1)$.

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Dimension Theorem

Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, $Q \in D$ be a dyadic cube, and $k \geq 1$ be an integer.

- Denote by $P_{Q,k}(\mu)$ the space of polynomials on Q which have degree less than k.

Theorem

$$\dim L^2_{Q,k} = \sum_{Q' \in C(Q)} \dim P_{Q',k}(\mu) - \dim P_{Q,k}(\mu).$$

- This extends to the additional moment vanishing conditions; each additional orthogonal monomial reduces the total dimension by 1.

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Classification

Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, $Q \in D$ be a dyadic cube, and $k \geq 1$ be an integer.

- The dimension theorem reduces the task of finding $\dim L^2_{Q,k}$ to finding $\dim P_{Q,k}(\mu)$.
- Equivalently, this is the task of finding all linear dependencies in $L^2(\mu)$ among the monomials of degree less than k.
- A monomial basis for some fixed k can be found by brute force using Gram-Schmidt.
- A more elegant solution uses techniques from algebraic geometry to find the set of all polynomials which vanish on the support of μ.

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Almost-Orthogonality

- A set $\{\phi_\gamma\}_{\gamma\in\Gamma} \subset L^2(\mu)$ is almost-orthogonal with AO-constant $0 \leq B$ if for all finite $F \in \Gamma$ and real numbers λ_γ we have

$$\left\|\sum_{\gamma\in F} \lambda_\gamma \phi_\gamma\right\|_2 \leq B \left(\sum_{\gamma\in F} |\lambda_\gamma|^2\right)^{1/2}.$$

- Suppose that $\{\phi_\gamma\}_{\gamma\in\Gamma}$ is an orthonormal basis for $L^2(\mu)$ and $\{\phi^*_\gamma\}_{\gamma\in\Gamma}$ is a family such that $\{\phi_\gamma - \phi^*_\gamma\}_{\gamma\in\Gamma}$ is almost-orthogonal with AO-constant δ. Then for any $f \in L^2(\mu)$ the series $\sum_{\gamma\in\Gamma}\langle f, \phi^*_\gamma\rangle\phi^*_\gamma$ converges to some $\widetilde{f} \in L^2(\mu)$ and $\|f - \widetilde{f}\|_2 \leq \delta(2+\delta)\|f\|_2$.

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Notion of Perturbation

Let μ be a locally-finite positive Borel measure on $\mathbb{R}$.

- Choose $0 \leq \eta < \frac{1}{2}$ and a dyadic grid D.
- For each $I \in D$, choose η_I such that $0 \leq |\eta_I| \leq \eta$.
- Define I^* to be the translation of I by $\eta_I \cdot \text{length}(I)$.
- Define h^{I^*} to be the Haar wavelet for I^* in $L^2(\mu)$.

Theorem

If μ is doubling then $\{h^I - h^{I^*}\}_{I \in D}$ is almost orthogonal with AO-constant at most $C\eta^p$ with constants $C, p > 0$ that depend only on μ.

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Theorem

If μ is doubling then $\{h^I - h^{I^*}\}_{I \in D}$ is almost orthogonal with AO-constant at most $C\eta^p$ with constants $C, p > 0$ that depend only on μ.

Stability Theorem

Proof structure:

- Goal: show that for all $I \in D$
$$\sum_{J \in D} \left| \langle h^I - h^{I^*}, h^J \rangle \right| \leq C\eta^p$$
and similarly with I and J reversed.
- For fixed I, J, most $\langle h^{I^*}, h^J \rangle = 0$. Remaining inner products reduce to ratios of measures of intervals.
- Both sums over all I and J converge with bound at most
$$\frac{36k^5}{(k^2-1)(k-1)}\eta^{\log_2\left(\frac{k^{2-\frac{1}{c_k}}}{k^2-1}\right)} + \frac{18\sqrt{2}k^3}{k-\sqrt{k^2-1}}\eta^{\frac{1}{2c_k}\log_2\left(\frac{k^2}{k^2-1}\right)} + \left(4k^4 + \frac{10k^7}{k^2-1}\right)\eta^{\log_2\left(\frac{k^2}{k^2-1}\right)}.$$
- By the Schur test, $\{h^I - h^{I^*}\}_{I \in D}$ is almost-orthogonal with AO-constant at most $C\eta^p$.

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Stability for Alpert Wavelets

Problems:

- Alpert wavelets are underdetermined.
- Given $Q \in D$, we need to match wavelets on Q to wavelets Q^* to use almost-orthogonality.
- There is no "obvious" way to take a function in $L^2_{Q,k}(\mu)$ and relate it to a function in $L^2_{Q^*,k}(\mu)$.

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Stability for Alpert Wavelets

Solution:

- Choose an ordering on the children of a dyadic cube and the N_k monomials in $\mathbb{R}^n$ of degree less than k.
- Let S be the ordered tuple containing the restrictions of the monomials to Q, then to Q_1, Q_2, ..., Q_{2^n}.
- Apply Gram-Schmidt to S.
  - First N_k outputs are an orthonormal basis for $P_{Q,k}(\mu)$.
  - Final N_k outputs are all zero.
  - Remaining $(2^n - 1)N_k$ outputs are an orthonormal basis for $L^2_{Q,k}(\mu)$.
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Let μ be a non-doubling measure

- For η arbitrarily small, find an interval I where $\mu(I^*) \gg \mu(I)$.
- Find a function near I where projections onto h^I and h^{I^*} are far apart.
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Example

Let μ be the measure with point masses at $\frac{1}{4}$ and $\frac{3}{4}$ and no mass elsewhere.

- Let D be the standard dyadic grid.
- $L^2(\mu)$ is spanned by 1 and $h^{[0,1)}$.
- h^I is non-zero iff $\frac{1}{4} \in I_{\text{left}}$ and $\frac{3}{4} \in I_{\text{right}}$.
- If h^I is non-zero then $h^I = h^{[0,1)}$.
- For $\eta < \frac{1}{4}$, $h^{I^*} = h^{[0,1)}$ iff $I = [0,1)$.
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Partial Results

- If μ is doubling then the Haar basis is stable for any choice of grid.
- If μ contains a point mass then there are grids for which the Haar basis is unstable.
- If μ contains an open set of measure 0 then there are grids for which the Haar basis is unstable.
- Conjecture: μ is doubling if and only if the Haar basis is stable for any choice of grid.

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Summary

- Weighted Alpert wavelets are degenerate when μ is supported inside the solution set to a collection of polynomials.
- The Alpert spaces $L^2_{Q,k}(\mu)$ decompose into polynomial spaces $P_{Q,k}(\mu)$.
- Weighted Alpert wavelets are stable in doubling measures.
- Weighted Haar wavelets can be conditionally stable in non-doubling measures.
- Open question whether non-doubling measures can be unconditionally stable.