On the regularity of axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions

Evan Miller

University of British Columbia
Pacific Institute for the Mathematical Sciences
(Joint work with Stephen Gustafson and Tai-Peng Tsai)

Introduction

- In this talk, we will consider the incompressible Euler equation:

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0 \qquad (1)$$
$$\nabla \cdot u = 0. \qquad (2)$$

- Smooth solutions of the Euler equations are known to exist globally in time for smooth initial data in $\mathbb{R}^2$ [Wolibner (1933), Hölder (1933)]
- There are also global smooth solutions in three dimensions in the presence of axial symmetry, without swirl [Ukhovskii and Yudovich (1968), X. Saint-Raymond (1994), Danchin (2007)].

One of the most central results for the regularity of solutions to the Euler equation in three and higher dimensions is the Beale-Kato-Majda criterion, which states that if a smooth solution to the Euler equation blows up in finite-time $T_{max} < +\infty$, then

$$\int_0^{T_{max}} \|\omega(\cdot, t)\|_{L^\infty} \, \mathrm{d}t = +\infty. \tag{3}$$

The setup

- We will consider axisymmetric, swirl-free solutions of the Euler equation, which have the form

$$u(x,t) = u_r(r,z,t)e_r + u_z(r,z,t)e_z \tag{4}$$

- The coordinates in this case are given by

$$r = \sqrt{x_1^2 + ... + x_{d-1}^2} \tag{5}$$
$$z = x_d \tag{6}$$
$$e_r = \frac{1}{r}(x_1, ..., x_{d-1}, 0) \tag{7}$$
$$e_z = e_d \tag{8}$$
$$k = d - 2. \tag{9}$$

The evolution equation

- The divergence free constraint can now be expressed by

$$\nabla \cdot u = \partial_r u_r + \frac{k}{r} u_r + \partial_z u_z = 0. \tag{10}$$

- The scalar vorticity is given by

$$\omega = \partial_r u_z - \partial_z u_r. \tag{11}$$

- The evolution equation for the vorticity is given by

$$\partial_t \omega + (u \cdot \nabla)\omega - \frac{k}{r} u_r \omega = 0. \tag{12}$$

- This results in the quantity $\frac{\omega}{r^k}$ being transported by the flow.

$$(\partial_t + u \cdot \nabla)\frac{\omega}{r^k} = 0. \tag{13}$$

Singular Structure

- This leads to a potential mechanism for singularity formation that is ruled out for smooth solutions in three dimensions.
- If $u \in C^2\left(\mathbb{R}^d\right)$, then $\frac{\omega}{r} \in L^\infty$.
- There is global regularity for smooth, axisymmetric solutions of the Euler equation in three dimensions.
- However, for rougher solutions, where $\frac{\omega^0}{r}$ is unbounded, Elgindi proved that there can be finite-time blowup in this setup.
- In $\mathbb{R}^4$, even for Schwartz class initial data, $\frac{\omega^0}{r^2}$ may be unbounded, so there is no barrier to finite-time blowup.

Higher dimensional dynamics

- Even when $\frac{\omega^0}{r^k}$ is bounded in higher dimensions, this cannot immediately lead to global regularity by the standard methods in three dimensions.
- In general, if $\frac{\omega^0}{r^k}$ is bounded and the vorticity is compactly supported, the radius of this support has bounded growth of the form
$$\frac{\mathrm{d}R}{\mathrm{d}t} \leq C(\omega^0) R^{\frac{d}{2}-1}. \tag{14}$$
- For $d = 3$, this gives quadratic growth in time, as proved by Choi and Jeong (2021).
- When $d = 4$, this gives exponential growth, and for $d \geq 5$, this bound cannot rule out finite-time blowup.

The potential for growth

- Even in the case where $\frac{\omega^0}{r^k}$ is bounded, it is not readily apparent that there is global regularity in sufficiently high dimensions.
- There could still be growth if the flow carries points out to infinity in finite-time.
- Even when $\frac{\omega^0}{r^k}$ is bounded, this only provides control on ω when there is some control on the stretching in the radial direction.

Key growth estimates

$$\omega\left(X(r,z,t)\right) = \frac{\omega^0(r,z)}{r^k} X_r(r,z,t)^k \tag{15}$$

$$X_r(r,z,t) \leq r + \int_0^t \|u_r^+(\cdot,\tau)\|_{L^\infty} \,\mathrm{d}\tau. \tag{16}$$

$$|\omega(X(r,z,t),t)| \leq \frac{|\omega^0(r,z)|}{r^k} \left(r + \int_0^t \|u_r^+(\cdot,\tau)\|_{L^\infty} \,\mathrm{d}\tau\right)^k \tag{17}$$

$$= |\omega^0(r,z)| \left(1 + \frac{1}{r}\int_0^t \|u_r^+(\cdot,\tau)\|_{L^\infty} \,\mathrm{d}\tau\right)^k \tag{18}$$

Key growth estimates (cont.)

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \max\left(\left\|\omega^0\right\|_{L^\infty(\mathcal{C}_R^c)}, R^k\left\|\frac{\omega^0}{r^k}\right\|_{L^\infty(\mathcal{C}_R)}\right) \left(1+\frac{1}{R}\int_0^t \|u_r^+(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau\right)^k. \quad (19)$$

$$\|\omega(\cdot,t)\|_{L^1} \leq \left(\left\|\omega^0\right\|_{L^1(\mathcal{C}_R^c)} + R^k\left\|\frac{\omega^0}{r^k}\right\|_{L^1(\mathcal{C}_R)}\right) \left(1+\frac{1}{R}\int_0^t \|u_r^+(\cdot,\tau)\|_{L^\infty}\,\mathrm{d}\tau\right)^k. \quad (20)$$

$$\|u_r\|_{L^\infty} \leq C_d\left\|\frac{\omega}{r^k}\right\|_{L^\infty}^{\frac{1}{2}} \|\omega\|_{L^1}^{\frac{1}{2}}. \quad (21)$$

The essential lemma

Lemma

Suppose u is an axisymmetric, swirl-free solution of the Euler equation, and that $\frac{\omega^0}{r^k} \in L^1 \cap L^\infty$. Then for all $R > 0$,

$$\|u_r(\cdot,t)\|_{L^\infty} \leq C_d \left\| \frac{\omega^0}{r^k} \right\|_{L^\infty}^{\frac{1}{2}} \left(\left\|\omega^0\right\|_{L^1(\mathcal{C}_R^c)} + R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^1(\mathcal{C}_R)} \right)^{\frac{1}{2}} \left(1 + \frac{1}{R} \int_0^t \|u_r(\cdot,\tau)\|_{L^\infty} \, \mathrm{d}\tau \right)^{\frac{k}{2}}. \quad (22)$$

Note that if

$$f(t) = 1 + \frac{1}{R} \int_0^t \|u_r(\cdot,\tau)\|_{L^\infty} \, \mathrm{d}\tau, \quad (23)$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} \leq \mu f^{\frac{k}{2}}. \quad (24)$$

Global regularity in three dimensions

Theorem

Suppose u^0 is axisymmetric and swirl-free and $d = 3$. Then there is a global smooth solution of the Euler equation, with for all $0 \leq t < +\infty$,

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \max\left(\left\|\omega^0\right\|_{L^\infty(\mathcal{C}_R^c)}, R\left\|\frac{\omega^0}{r}\right\|_{L^\infty(\mathcal{C}_R)}\right)\left(1+\frac{1}{2}\mu t\right)^2, \tag{25}$$

This bound is proven by Choi and Jeong with additional condition of compact support, but removing this condition does not require a fundamental change in methods.

Global regularity in four dimensions

Theorem

Suppose u^0 is axisymmetric and swirl-free, $d = 4$, and $\frac{\omega^0}{r^2} \in L^1 \cap L^\infty$. Then there is a global smooth solution of the Euler equation, with for all $0 \leq t < +\infty$,

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \max\left(\left\|\omega^0\right\|_{L^\infty(\mathcal{C}_R^c)}, R^2 \left\| \frac{\omega^0}{r^2} \right\|_{L^\infty(\mathcal{C}_R)} \right) \exp(2\mu t), \tag{26}$$

Global regularity in four dimensions

Theorem

Suppose u^0 is axisymmetric and swirl-free, $d \geq 5$, and $\frac{\omega^0}{r^k} \in L^1 \cap L^\infty$. Then for all $0 \leq t < T_{max}$,

$$\|\omega(\cdot,t)\|_{L^\infty} \leq \frac{\max\left(\left\|\omega^0\right\|_{L^\infty}, R^k \left\|\frac{\omega^0}{r^k}\right\|_{L^\infty}\right)}{(1-\alpha t)^{\frac{d-2}{d-4}}}, \tag{27}$$

Regularity criteria

Theorem

Suppose u is a axisymmetric, swirl-free smooth solution of the Euler equation on $\mathbb{R}^d, d \geq 5$ and that $\frac{\omega^0}{r^k} \in L^1 \cap L^\infty$. Then for all $0 \leq t < T_{max}$,

$$\|\omega(\cdot,t)\|_{L^1(\mathbb{R}^d)} \geq \frac{C_d\left[\omega^0\right]}{(T_{max}-t)^{\frac{2(d-2)}{d-4}}}, \tag{28}$$

and

$$\int_0^{T_{max}} \left\|u_r^+(\cdot,t)\right\| \mathrm{d}t = +\infty. \tag{29}$$

Conjecture

Conjecture

There exists a $d_0 \geq 5$, *such that for all* $d \geq d_0$, *if* $u^0 \in H^s_{as\&df}\left(\mathbb{R}^d\right), s > 1 + \frac{d}{2}$, *such that the associated vorticity* ω^0 *is odd in* z, *and for all* $z > 0, \omega^0(r,z) \geq 0$, *and the smooth solution* $u \in C\left([0,T_{max}); H^s_{df}\left(\mathbb{R}^d\right)\right) \cap C^1\left([0,T_{max}), H^{s-1}_{df}\left(\mathbb{R}^d\right)\right)$ *with initial data* u^0 *blows up up in finite-time,* $T_{max} < +\infty$.

Heuristic arguments based on scaling suggest that finite-time blowup should occur for $d \geq 6$, but proving this would require much finer knowledge about the behaviour of solutions.

Blowup Scenario

- Our possible scenario for finite-time blowup involves a vorticity that is odd in z and positive for $z > 0$.
- This geometric setup—axisymmetric, swirl-free, with a sign condition on the vorticity, has substantial precedent for finite or infinite-time blowup.
- In this scenario for the three dimensional Euler equation, Elgindi (2021) proved finite-time blowup for non-smooth solutions of the 3D Euler equation, and Choi and Jeong (2021) proved the proved the blowup at infinity of the vorticity with

$$\|\omega(\cdot,t)\|_{L^\infty} \geq C(1+t)^{\frac{1}{15}-\epsilon}. \tag{30}$$

- To justify this blowup scenario, we will consider the limiting equation for the vorticity when we take $d \to \infty$, and prove blowup in this case.

The stream function

- There is a stream function formulation of the vorticity equation in four and higher dimensions as well:

$$u_z = -\frac{1}{k}\partial_r \tilde{\psi} - \frac{\tilde{\psi}}{r} \tag{31}$$

$$u_r = \frac{1}{k}\partial_z \tilde{\psi} \tag{32}$$

$$\left(-\frac{1}{k}\partial_z^2 - \frac{1}{k}\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\right)\tilde{\psi} = \omega. \tag{33}$$

- Taking the infinite dimensional limit of this stream function yields: Taking the formal limit $k \to +\infty$, we obtain the equations

$$\omega = -\frac{1}{r}\partial_r \tilde{\psi} + \frac{1}{r^2}\tilde{\psi} \tag{34}$$

$$= -\partial_r \left(\frac{\tilde{\psi}}{r}\right). \tag{35}$$

The infinite dimensional vorticity equation

- Taking the infinite-dimensional limit of the vorticity evolution equation gives us

$$\partial_t \omega - \frac{\tilde{\psi}}{r} \partial_z \omega - \omega \partial_z \left(\frac{\tilde{\psi}}{r} \right) = 0. \tag{36}$$

- Making the substitution $\phi = \frac{\tilde{\psi}}{r}$, we obtain the infinite-dimensional vorticity equation.

$$\partial_t \omega + \phi \partial_z \omega + \omega \partial_z \phi = 0 \tag{37}$$

$$\partial_r \phi = \omega. \tag{38}$$

- This equation exhibits finite-time blowup for a very wide range of data, and this blowup is of a Burgers shock type.

Relationship with the 1D Burgers equation

- In fact, ω is a solution of the infinite-dimensional vorticity equation if and only if ϕ is a solution to 1D Burgers,

$$\partial_t \phi + \phi \partial_z \phi = 0. \tag{39}$$

- This is straightforward to prove. Suppose

$$\partial_t \phi + \phi \partial_z \phi = 0. \tag{40}$$

Then we can see that

$$\begin{aligned} \partial_t \omega + \phi \partial_z \omega + \omega \partial_z \phi &= \partial_t \partial_r \phi + \phi \partial_z \partial_r \phi + \partial_r \phi \partial_z \phi && (41) \\ &= \partial_r \left(\partial_t \phi + \phi \partial_z \phi \right) && (42) \\ &= 0. && (43) \end{aligned}$$

Blowup result for the infinite-dimensional vorticity equation

Theorem

For all initial data ω^0, there exists a unique strong solution ω to the infinite-dimensional vorticity equation. If $\partial_z \phi^0(r,z) \geq 0$ for all $r \in \mathbb{R}^+, z \in \mathbb{R}$, then there is a global smooth solution. Otherwise, there is finite-time blowup with

$$T_{max} = \frac{1}{-\inf_{\substack{r\in\mathbb{R}^+ \\ z\in\mathbb{R}}} \partial_z \phi^0(r,z)}. \tag{44}$$

This solution of the infinite-dimensional vorticity equation is given by

$$\omega(r,z,t) = \frac{\omega^0(r,h(r,z,t))}{1+t\partial_y\phi^0(r,h(r,z,t))}, \tag{45}$$

where $h(r,z,t)$ is the back-to-labels map.

Theorem

Suppose u is an axisymmetric, swirl free solution of the Euler equation on $\mathbb{R}^d, d \geq 3$. Suppose that $\omega(r,-z) = -\omega(r,z)$ and for all $z > 0$, $\omega(r,z) \geq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \int_0^\infty \omega(r,z,t)\,\mathrm{d}r\,\mathrm{d}z = 0 \tag{46}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \int_0^\infty r^{d-1}\omega(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \geq 0 \tag{47}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \int_0^\infty z\omega(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \leq 0. \tag{48}$$

- Iftime, Sideris, and Gamblin (1999) used similar arguments when $d = 2$, as did Choi and Jeong (2021) when $d = 3$.