- Given equations $ay''+by'+cy=F$, we can use educated guesses to find a particular solution when F is in certain "popular" forms.

Ex: $2y''+3y'-y=e^{2t}$; find y_p.

Sol$^{\underline{n}}$: It is somewhat useful to think of

$y_p(t)=Ae^{2t}$ for some $A\in\mathbb{R}$.

Notice, $y_p'=2Ae^{2t}$

$y_p''=4Ae^{2t}$

So, our eq^{n} becomes

$$8Ae^{2t}+6Ae^{2t}-Ae^{2t}=13Ae^{2t}=e^{2t}$$

So, $A=1/13$ giving $y_p(t)=\frac{1}{13}e^{2t}$

- $2r^2+3r-1=0$ $r_{1,2}=\frac{-3\pm\sqrt{17}}{4}$; $y_H(t)=Ae^{r_1t}+Be^{r_2t}$

i.e. $y(t)=Ae^{r_1t}+Be^{r_2(t)}+\frac{1}{13}e^{2t}$ is our general sol$^{\underline{n}}$.

- Initial Conditions: If $y(0)=1$, $y'(0)=0$

then, $\bullet\ A+B+\frac{1}{13} = 1 \rightarrow A = \frac{12}{13} - B$

$\bullet\ r_1 A + r_2 B + \frac{2}{13} = 0$

$\rightarrow \frac{12 r_1}{13} - r_1 B + r_2 B + \frac{2}{13} = 0$

$\rightarrow (r_2 - r_1) B = -\frac{14}{13} \rightarrow B = \frac{14}{13(r_1 - r_2)}$

$\rightarrow A = \frac{12(r_1 - r_2) - 14}{13(r_1 - r_2)}$

$\bullet$ Form of $F(t)$	Choice of Y_p
(*) IF r is not a root of the characteristic poly, 1. Ae^{rt}	Be^{rt}
(*) $\pm i\mu$ are not roots of the characteristic poly 2. $A\cos(\mu t) + B\sin(\mu t)$	$C\cos(\mu t) + D\sin(\mu t)$
3. $P_n(t) = a_0 + a_1 t + \dots + a_n t^n$	$t^s(A_0 + A_1 t + \dots + A_n t^n)$ (*)
4. $P_n(t) e^{\alpha t} \begin{cases} \sin(\mu t) \\ \cos(\mu t) \end{cases}$	$t^s(A_0 + A_1 t + \dots + A_n t^n) e^{\alpha t} \cos \mu t$ (*) $+ t^s(B_0 + B_1 t + \dots + B_n t^n) e^{\alpha t} \sin \mu t$

(*) $s \in \{0,1,2\}$ and is the smallest value of s that ensures each term on the right of our table is not a solution of $ay'' + by' + cy = 0$

Ex: Find a particular Sol$^{\underline{n}}$ of

$$y'' + 4y' + 4y = te^{-2t}.$$

Sol$^{\underline{n}}$: $\bullet\ r^2 + 4r + 4 = (r+2)^2 \quad r = -2, \quad y_H = Ae^{-2t} + Bte^{-2t}$

We want to use $y_p(t) = (Ct+D)e^{-2t}$ but, Cte^{-2t} 3

is in the homog. sol^n y_H. We need $s=2$ to over come this,

$$y_p(t) = t^2(Ct+D)e^{-2t}$$
$$= (Ct^3+Dt^2)e^{-2t}$$

- $y_p' = (3Ct^2+2Dt)e^{-2t} - 2(Ct^3+Dt^2)e^{-2t}$
$$= \left(-2Ct^3+(3C-2D)t^2+2Dt\right)e^{-2t}$$

- $y_p'' = (-6Ct^2+(6C-4D)t+2D)e^{-2t}$
$$+ (4Ct^3+(4D-6C)t^2-4Dt)e^{-2t}$$
$$= \Big[\big(4Ct^3+(4D-12C)t^2+(6C-8D)t+2D\Big]e^{-2t}$$

we want

$$0t^3 + (\cancel{4D}-\cancel{12C}) - 8Dt^2 + 6C-\cancel{8D} \cdot t + 2D$$
$$+\cancel{12C}-8D \qquad\qquad + \cancel{8D}$$
$$-\cancel{4D}$$

$D=0$, $C = 1/6$.

i.e. $\boxed{y_p(t) = \frac{1}{6}t^3e^{-2t}}$.

Variation of Parameters

Consider $y'' + p(t)y' + q(t)y = F(t)$

- Suppose we have 2 sol$^{\underline{n}}$s y_1, y_2 of

$$y'' + p(t)y' + q(t)y = 0$$

Thinking of y_1, y_2 as basis vectors, we guess that

$$\boxed{y_p = u_1(t)y_1(t) + u_2(t)y_2(t)}$$

This gives: $y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$

- We will assume $(u_1', u_2') \cdot (y_1, y_2) = 0$

this gives $\boxed{y_p' = u_1 y_1' + u_2 y_2'}$

Differentiating some more:

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

So, $y_p'' + p(t)y_p' + q(t)y_p$

$$= u_1(\underbrace{y_1'' + py_1' + qy_1}_{0}) + u_2(\underbrace{y_2'' + py_2' + qy_2}_{0}) + u_1' y_1' + u_2' y_2' = F(t)$$

So,

Our Assumption:

$$\begin{cases} u_1' y_1' + u_2' y_2' = F(t) \\ u_1' y_1 + u_2' y_2 = 0 \end{cases}$$

By Kramer's Rule:

$$u_1' = -\frac{y_2 F}{W(y_1,y_2)} \qquad u_2' = \frac{y_1 F}{W(y_1,y_2)}$$

- $u_1(t) = -\int \frac{y_2 F}{W(y_1,y_2)}\,dt$
- $u_2(t) = \int \frac{y_1 F}{W(y_1,y_2)}\,dt$

Our particular solⁿ is

$$y_p(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$$

Ex: Solve $y'' - 2y' + y = 2e^{6t}$; $y(0) = 1$, $y'(0) = 0$

Solⁿ: By Variation of parameters

- $r^2 - 2r + 1 = 0$ $r = 1$ is a double root.

$$y_1 = e^t, \quad y_2 = te^t$$

$$y_H(t) = Ae^t + Bte^t$$

- $W(y_1,y_2) = \det\begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$

$$= e^{2t} + te^{2t} - te^{2t}$$

$$= e^{2t}$$

$$u_1 = -\int \frac{te^t \cdot 2e^{6t}}{e^{2t}}\,dt = -2\int te^{5t}\,dt. \qquad u = t \quad dv = e^{5t}$$

$$du = dt \quad v = \tfrac{1}{5}e^{5t}$$

$$= -2\left(\frac{t}{5}e^{5t} - \frac{1}{25}e^{5t}\right) = -\frac{2}{5}\left(t - \frac{1}{5}\right)e^{5t}$$

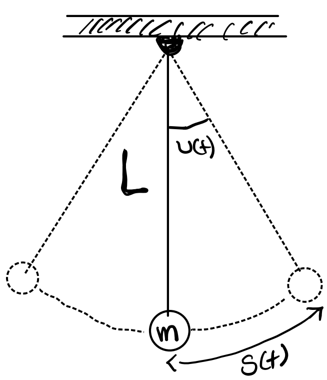
$$u_2 = \int \frac{e^t \cdot 2e^{6t}}{e^{2t}}\,dt = 2\int e^{5t}\,dt = \frac{2}{5}e^{5t}$$

So, $y_p = u_1y_1 + u_2y_2 = -\frac{2}{5}\left(t-\frac{1}{5}\right)e^{6t} + \frac{2}{5}te^{6t}$

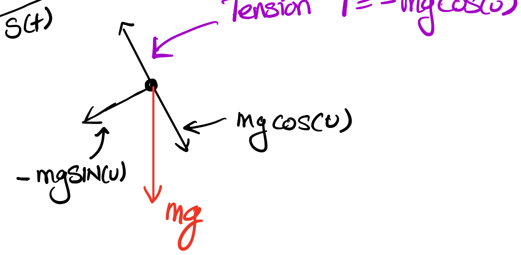
$$= \frac{2}{25}e^{6t}$$

THIS method will always work, But the integrals can be very tough. Undetermined Coefficients can be an un-looked for friend.

For Fun: Motion of a Simple Pendulum



- $\theta = 0$: equilibrium
- $u(t)$ is angular displacement

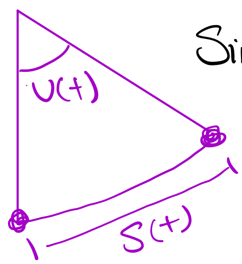


Adding all forces and using $F=ma$:

- $F = -mg\sin(u) = ma$

so $a = -g\sin(u)$. Now, $a(t) = \frac{d^2}{dt^2}s(t)$

where s is displacement from eq$^{\underline{m}}$, this is arclength along the circular arc the pendulum follows



Since we work in radians,

$$s(t) = L\,v(t)$$

giving $s''(t) = L\,v''(t)$

i.e. $v'' = \frac{1}{L} s'' = -\frac{g}{L}\sin(v)$

- $v'' + \frac{g}{L}\sin(v) = 0$

(*) If v is small, $\sin(v) \approx v$ $\left(\text{Recall } \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \right)$

we find a small oscillation model of angular motion

- $v'' + \frac{g}{L} v = 0$

$v(t_0) = v_0$; initial angular position
$v'(t_0) = v_1$; initial angular vel.

Notice: The characteristic polyⁿ is $r^2 + \frac{g}{L}$ and has Complex Roots

$$r_1 = \frac{\sqrt{g}}{\sqrt{L}} i, \quad r_2 = -\frac{\sqrt{g}}{\sqrt{L}} i$$

Solutions are

$$u_1(t) = \sin\left(\sqrt{\frac{g}{L}}\, t\right)$$

$$u_2(t) = \cos\left(\sqrt{\frac{g}{L}}\, t\right)$$

and the general sol^{n} is

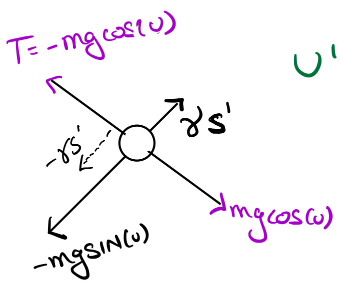
$$u(t) = A\sin\left(\sqrt{\frac{g}{L}}\, t\right) + B\cos\left(\sqrt{\frac{g}{L}}\, t\right)$$

So, the pendulum oscillates with a period

$$p = 2\pi\sqrt{\frac{L}{g}}.$$

Friction adds the force $F_{fric} = \gamma s'(t) = \frac{\gamma}{L} u'(t)$

our equation is modified to



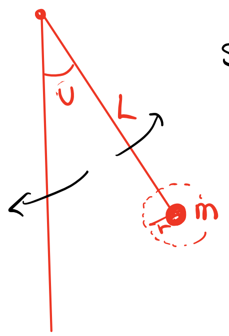
$$u'' + \frac{\gamma}{L} u' + \frac{g}{L} u = F(t)$$

where F is any external force acting along the arc.

This produces "damped", "critically damped" and "overdamped" oscillations.

- Damped: $0 < \gamma < 2\sqrt{gL}$
- Critical: $\gamma = 2\sqrt{gL}$
- Over Damped: $\gamma > 2\sqrt{gL}$

Animating a pendulum (FREE Oscillations)



Set $L=1$, $g=10$

Eq^n: $U'' + 10U = 0$

Solutions: $U(t) = A\sin(\sqrt{10}\,t) + B\cos(\sqrt{10}\,t)$

- $U(0) = 1/2$, $U'(0) = 0$
- $U(0) = 1/2 \rightarrow B = 1/2$.
- $U'(0) = \sqrt{10}A = 0 \rightarrow A = 0$.

$$U(t) = \frac{1}{2}\cos(\sqrt{10}\,t).$$

to animate, suppose tie is at $(0,0)$. The pendulum is a circle of radius $r>0$ about the centre point

$$C(t) = L\left(\cos\left(\frac{3\pi}{2}+U\right), \sin\left(\frac{3\pi}{2}+U\right)\right)$$

$\frac{3\pi}{2}$ is there since $U(t)$ is now deviation from the $-$ve y-axis.

$$= L\left(\cos\left(\frac{3\pi}{2}+\frac{\cos(\sqrt{10}\,t)}{2}\right), \sin\left(\frac{3\pi}{2}+\frac{\cos(\sqrt{10}\,t)}{2}\right)\right)$$

The line connecting $(0,0)$ to our pendulum is parametrized by

$$L(t) = w\,C(t) \quad \text{where } w \in [0,L].$$

In Desmos: For $L=1$:

Enter: $s = \cos\left(\frac{3\pi}{2} + \frac{1}{2}\cos(\sqrt{10}\,t)\right)$

$\ell = \sin\left(\frac{3\pi}{2} + \frac{1}{2}\cos(\sqrt{10}\,t)\right)$

where t is on a "SLIDER" from $t=0$ to $t=\frac{2\pi}{\sqrt{10}}$ step: 0.025.

Enter: (s,ℓ)

Enter: $y = \frac{\ell}{s}x \left\{ \frac{\ell}{\sqrt{s^2+\ell^2}} \le y \le 0 \right\}$

External Forces acting on a free oscillation.

- Assume Our External force is periodic and decays as

$$F(t) = e^{-\sqrt{10}t}\cos(\sqrt{10}t)$$

Use the method of U.D.C. to solve:

$$\begin{cases} u'' + 10u = F(t) \\ u(0) = 1/2, \\ u'(0) = 0 \end{cases}$$

if you can, animate the solution!

- $y_H(t) = A\cos(\sqrt{10}t) + B\sin(\sqrt{10}t)$
- $y_p(t) = e^{-\sqrt{10}t}(A\cos\sqrt{10}t + B\sin(\sqrt{10})t)$

$$y_p' = -\sqrt{10}\,y_p(t) + \sqrt{10}e^{-\sqrt{10}t}(B\cos(\sqrt{10}t) - A\sin(\sqrt{10}t))$$

$$y_p'' = -\sqrt{10}\,y_p' - 10e^{-\sqrt{10}t}(B\cos(\sqrt{10}t) - A\sin(\sqrt{10}t)) - 10e^{-\sqrt{10}t} \cdot (A\cos(\sqrt{10}t) + B\sin(\sqrt{10}t))$$

So $y_p'' + 10y_p$

$$= -\sqrt{10}\,y_p' - 10e^{-\sqrt{10}t}(B\cos(\sqrt{10}t) - A\sin(\sqrt{10}t))$$

$$= 10y_p - 20e^{-\sqrt{10}t}(B\cos(\sqrt{10}t) - A\sin(\sqrt{10}t))$$

$$= 10e^{-\sqrt{10}t}((A-2B)\cos(\sqrt{10}t) + (B-2A)\sin(\sqrt{10}t))$$

$$= e^{-\sqrt{10}t}\cos(\sqrt{10}t)$$

So $\begin{matrix} A - 2B = 1/10 \\ B - 2A = 0 \end{matrix} \longrightarrow \begin{matrix} 2A - 4B = 1/5 \\ B - 2A = 0 \end{matrix}$

$\hookrightarrow -3B = 1/5 \rightarrow B = -1/15$

$\rightarrow A = -1/30$

$$\rightarrow y_p(t) = -\frac{1}{30} e^{-\sqrt{10}t}\left(\cos(\sqrt{10}t) - 2\sin(\sqrt{10}t)\right)$$

- $y(t) = y_p(t) + C\cos(\sqrt{10}t) + D\sin(\sqrt{10}t)$

- $y(0) = -\frac{1}{30} + C = 1/2$; $C = \frac{1}{2} + \frac{1}{30} = \frac{16}{30}$

- $y'(0) = y_p'(0) + \sqrt{10}D = 0$

  So $D = \frac{-y_p'(0)}{\sqrt{10}} = 1/30.$

$$* \; y(t) = -\frac{1}{30} e^{-\sqrt{10}t}\left(\cos(\sqrt{10}t) - 2\sin(\sqrt{10}t)\right) + \frac{16}{30}\cos(\sqrt{10}t) + \frac{1}{30}\sin(\sqrt{10}t)$$

Can you animate this one?