2208 L7

Repeated Roots

Consider $ay''+by'+cy=0$ with $b^2-2ac=0$;

i.e. $r=-\frac{b}{2a}$ is the only root of $ar^2+br+c=0$

Of course, $y_1=e^{rt}$ is a solution of the eq^{n}. But, we don't have a second!

WE GUESS that $y_2(t)=v(t)e^{rt}$ for some nice function $v(t)$.

Then, • $y_2' = v'e^{rt}+rve^{rt}$

• $y_2'' = v''e^{rt}+2rv'e^{rt}+r^2ve^{rt}$.

Putting these in our equation and collecting terms,

$$\begin{aligned} ay_2''+by_2'+cy_2 &= v\underbrace{(ar^2+br+c)}_{=0!}e^{rt} \\ &\quad+(av''+(2ar+b)v')e^{rt} \\ &= av''e^{rt} \quad \text{since } 2ar+b=-b+b=0. \end{aligned}$$

So, $y_2=ve^{rt}$ is a sol^{n} if

$$v''=0$$

i.e. $v(t)=At+B$ for some A,B.

For simplicity, choose $v(t)=t$.

This gives us two solutions of our equation

$$y_1 = e^{rt}, \quad y_2 = te^{rt}.$$

Checking $W(e^{rt}, te^{rt})$

$$= \det\begin{pmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{pmatrix}$$

$$= e^{2rt} + rte^{2rt} - tre^{2rt}$$

$$= e^{2rt} \neq 0 \text{ for any } t \in \mathbb{R}.$$

i.e. y_1, y_2 form a FSS for our eqn!

Going back to our Spring.

$$my'' + \gamma y' + ky = 0$$

The discriminant of the Ch. Polyn is

$$\gamma^2 - 4mk = (\gamma - 2\sqrt{mk})(\gamma + 2\sqrt{mk})$$

Since $\gamma, m, k \geq 0$, there is one value of γ s.t. $\gamma^2 - 4mk = 0$; $\gamma^* = 2\sqrt{mk}$

(A) • $\gamma^2 - 4mk > 0$: $y = Ae^{r_1 t} + Be^{r_2 t}$

(B) • $\gamma^2 - 4mk = 0$: $y = Ae^{-\frac{\gamma}{2m}t} + Bte^{-\frac{\gamma}{2m}t}$

Ⓒ. $\gamma^2 - 4mk < 0$: $y = (A\cos(\mu t) + B\sin(\mu t))e^{-\frac{\gamma t}{2m}}$

where $\mu^2 = \frac{4km - \gamma^2}{2m}$

3

Ⓐ: $r_j = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

Since $\gamma, m, k > 0$, $\gamma^2 - 4mk < \gamma^2$ so

$$\sqrt{\gamma^2 - 4mk} < \sqrt{\gamma^2} = \gamma$$

We see $r_1, r_2 < 0$.

i.e. $\lim_{t \to \infty} y(t) = 0$ and there is no oscillation.

Ⓑ $y(t) = (A + Bt)e^{\frac{-\gamma}{2m}t} \longrightarrow 0$ as $t \to \infty$ by L'Hospital!

no oscillation but might overshoot

Ⓒ Since $|y(t)| \leq (|A| + |B|)e^{-\frac{\gamma}{2m}t}$ we see

$y(t) \longrightarrow 0$ as $t \to \infty$. lots of oscillation!

- Undamped, free oscillation

  $\gamma = 0$

- Damped Oscillations

  $0 < \gamma < 2\sqrt{km}$

- Critically Damped

  $\gamma = 2\sqrt{km}$

- Over damped

$$\gamma > 2\sqrt{km}$$

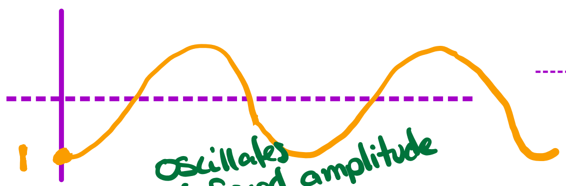
Examples

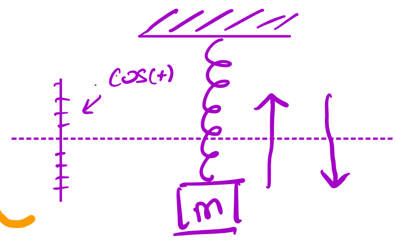
- Initial Conditions: $y(0) = 1$, $y'(0) = 0$

I

$\gamma = 0, m = 1, k = 1$: $y'' + y = 0$; $r^2 + 1 = 0$ $r = \pm i$ $\quad \gamma = 0$

$y(t) = \cos(t)$





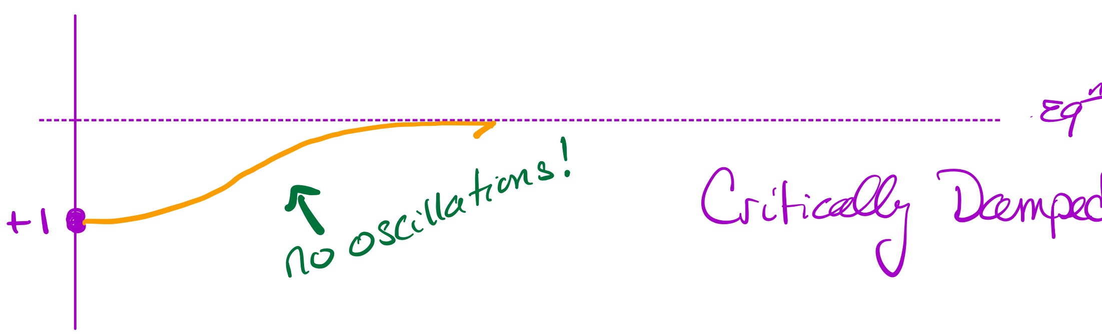
undamped $\gamma = 0$

II

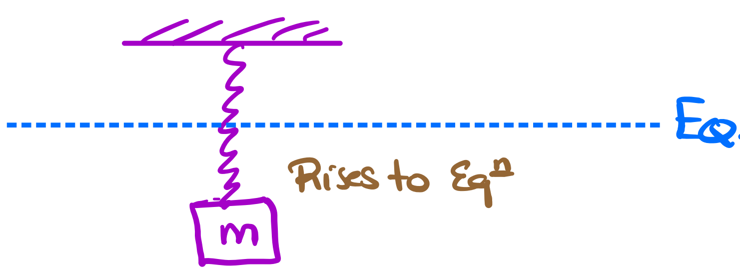
$\gamma = 2, m = 1, k = 1$ $\quad \gamma = 2\sqrt{mk}$

$y'' + 2y' + y = 0$; $y(t) = (1+t)e^{-t}$.

critically damped



Critically Damped!

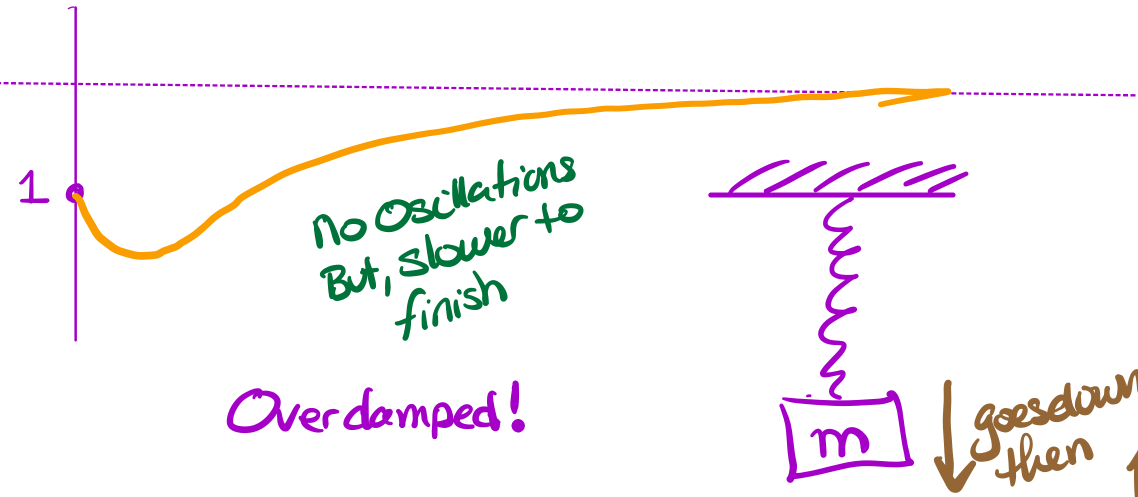


III

$\gamma = 4, m = 1, k = 1$ $\quad \gamma > 2\sqrt{mk}$

$y'' + 4y' + y = 0$, $\quad y(t) = \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)e^{r_1 t} + \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)e^{r_2 t}$

where $r_1 = -2 - \sqrt{3}$, $r_2 = -2 + \sqrt{3}$



Overdamped!

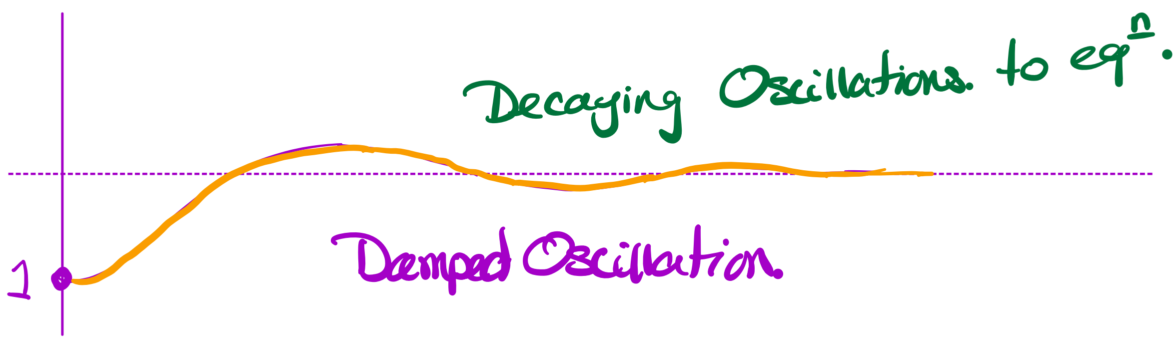
goes down then ↑ slowly to Eq"

(iv) $m=1,\ \gamma=1,\ k=1$ $\quad 0<\gamma<2\sqrt{mk}$

$y''+y'+y=0,\quad r^2+r+1=0$

$r=\dfrac{-1\pm\sqrt{3}i}{2}$

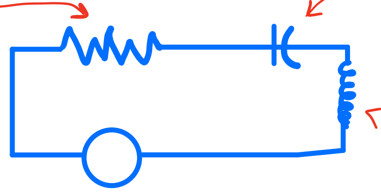
$$y(t)=e^{-\frac{t}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}t\right)+\frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$



Electric Circuits:

Capacitance C

Resistance R in Ohms



Simple Circuit

Inductance L in Henrys (H)

External Voltage E(t). in Volts

or "impressed"

Current in the circuit is $I(t)$ measured in Amps.

The Charge on the Capacitor at time t is

$Q(t)$ C (coulombs)

$$(*) \quad I(t) = \frac{dQ}{dt}$$

Using Kirchhoff's Law: THE Impressed Voltage is equal to the sum of Voltage drops in the rest of the circuit.

THE Drops:
- Resistor is $RI(t)$
- Capacitor is $\frac{Q}{C}$
- Inductor is $L\frac{dI}{dt}$

So, $$RI + \frac{Q}{C} + L\frac{dI}{dt} = E(t)$$

OR in Q: $$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

Initial Conditions: $Q(t_0) = Q_0$; Initial Charge on cap.
$Q'(t_0) = Q_1$; Initial Current

differentiating again:

$$L \frac{d^2}{dt^2}\left(\frac{d}{dt}Q\right) + R\frac{d}{dt}\left(\frac{d}{dt}Q\right) + \frac{1}{C}\frac{d}{dt}Q = E'(t)$$

OR

$$\boxed{L I'' + R I' + \frac{1}{C} I = E'(t).}$$

Initial Conditions

$$I(t_0) = I_0 \; ; \; I'(t_0) = I_1$$

Initial Current

harder to measure

go back to

$$I_1 = I'(t_0) = \frac{E(t_0) - \frac{Q(t_0)}{C} - R I_0}{L}$$

We see in these applications, external forces or Voltages give rise to Inhomogeneous equations of the form

$$ay'' + by' + cy = F(t)$$

maybe not 0 !

Suppose now we found a solution w. Then, given any sol^{n} z of

$$ay'' + by' + cy = 0$$

we find $y(t) = z(t) + w(t)$, this also satisfies our original eqⁿ.

Strategy to Solving $\begin{cases} ay'' + by' + cy = F(t) \\ y(t_0) = y_0, \; y'(t_0) = y_1 \end{cases}$

1. Find the general solution of $ay'' + by' + cy = 0$.

Call this the Gen. Solⁿ y_H of the related homogeneous problem.

2. Find, using any method you can a "Particular Solution" y_p of $\{ ay'' + by' + cy = F$

3. Then, $y(t) = y_H(t) + y_p(t)$ is the general solution of our original problem.

We know how to do 1!
Next time, the method of Undet. Coeffs.