Separable Equations

So far: $y' + B(t)y = D(t)$

Next, we consider a special class of possibly non-linear equations.

- $y' = f(t,y) = \psi(t)\varphi(y)$ for $t \in [a,b]$ — the variables are separated

Solution: Rewrite as

$$\frac{1}{\varphi(y)}\frac{dy}{dt} = \psi(t)$$

If $\frac{d}{ds}Q(s) = \frac{1}{\varphi(s)}$ then our equation becomes

$$\frac{d}{dt}\left[Q(y)\right] = \psi(t)$$

So, integrating we find

$$Q(y) = \int_a^t \psi(s)\,ds + C$$

$$\longrightarrow \quad y(t) = Q^{-1}\left[\int_a^t \psi(s)\,ds + C\right]$$

Ex: $y' = ky(q-y)$

- $\frac{1}{y(q-y)}y' = k$ $\quad (0<y<q)$

i.e. $\left[\frac{1}{9-y} + \frac{1}{y}\right] dy = 9k$

i.e. $-\ln \frac{1}{9-y} + \ln y = 9kt + C$

i.e. $\ln \frac{y}{9-y} = 9kt + C$

$\longrightarrow \frac{y}{9-y} = Ae^{9kt}$

$$\boxed{y = \frac{9Ae^{9kt}}{1+Ae^{9kt}}}$$

this is a 1-parameter family of solutions

$\frac{9A}{1+A} = y(0)$

$\frac{y(0)}{9} = \frac{A}{1+A}$

$A\left(1 - \frac{y(0)}{9}\right) = \frac{y(0)}{9}$

$A = \frac{y(0)/9}{1 - y(0)/9}$

Ex: Find a solⁿ of

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}$$

that passes through $(0,1)$.

Solⁿ: $(4+y^3)\,dy = (4x - x^3)\,dx$

$\longrightarrow 4y + \frac{1}{4}y^4 = 2x^2 - \frac{1}{4}x^4 + C$

i.e. $y^4 + 16y + x^4 - 8x^2 = C$

Notice, $(0,1)$ is on this locus if $x=0$ & $C=17$.

plot some pics.

Ideas on drawing pics.

- Phase Plane : $(t, y(t))$

* vectorfield lines

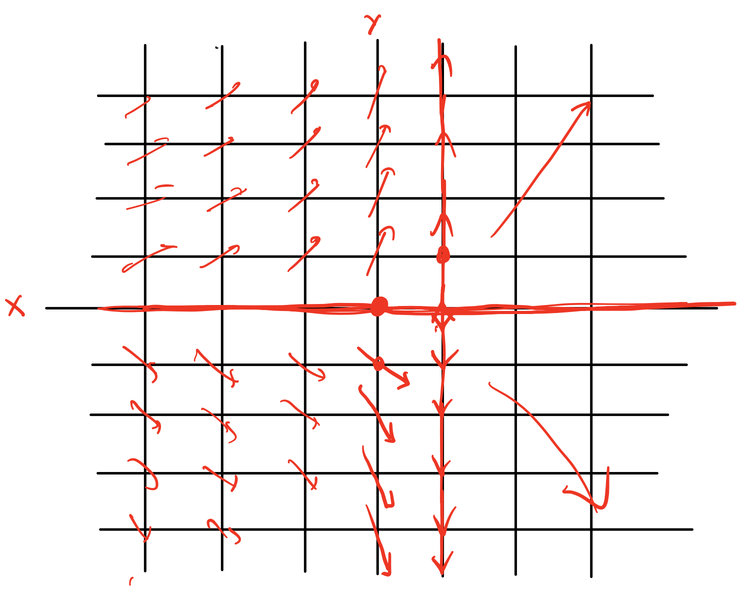
$y' = f(t,y)$ • Given (t,y), $f(t,y) \in \mathbb{R}$ and gives the slope of our sol$^{\underline{n}}$

Ex: $y' = \dfrac{y}{(x-1)^2}$

• $\ln(y) = \dfrac{-1}{x-1} + C$

$y = C e^{-\frac{1}{x-1}}$

for any $C \in \mathbb{R}$.



Ex: A Mixing problem. at $t=0$, Q_0 ~~kg~~ of salt in tank. 100L

Saltwater at $\frac{1}{4}$ kg/L (r L/min) enters a mixing tank

and flows out at r L/min

- THE IVP for this system:

Rate of change of salt in tank:

- $Q(t)$: kg of salt at time t mins
- $Q'(t) = \text{Rate in} - \text{Rate out}$

$$= \frac{r}{4} - \frac{rQ}{100}$$

i.e. $Q'(t) + \frac{rQ}{100} = \frac{r}{4}$

$Q(0) = Q_0$

$$\mu(t) = \exp\left(\frac{rt}{100}\right)$$

$$Q(t) = e^{-\frac{rt}{100}} \int \frac{r}{4} e^{+\frac{rt}{100}} dt$$

$$= \frac{r}{4} e^{-rt/100} \cdot \left[\left(+\frac{100}{r}\right) e^{+rt/100} + c \right]$$

$$= 25 + 25re^{-rt/1000}$$

<u>NON-LINEAR Vs. LINEAR.</u>

- We proved existence of solutions to

$$y' + B(t)y = D$$

and uniquess of the general sol^{n}.

More, given any initial condition

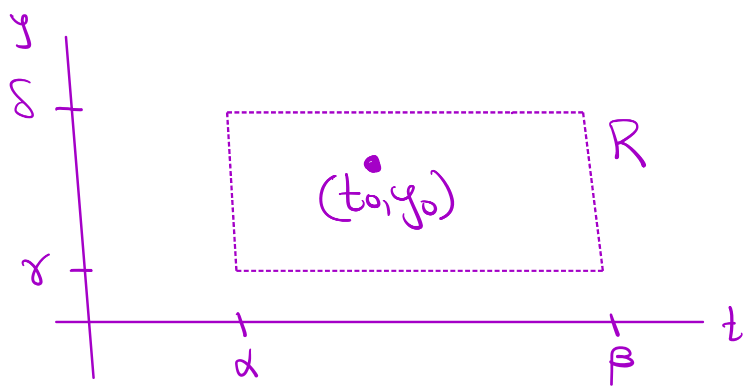
Say $y(t_0) = Y_0$, then there is exactly 1 solution of the problem.

Non-linear Equations are not so simple!

Consider now $y' = f(t,y)$ where $\alpha < t < \beta$, $\gamma < y < \delta$ contains the point (t_0, y_0)

$\downarrow$

$y(t_0) = y_0$



- If $f(t,y)$ & $\frac{\partial f}{\partial y}(t,y)$ are continuous on R

then there is a unique solution $y(t)$ of the problem

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \quad \text{for } t \in (t_0 - h, t_0 + h)$$

Ex: If $y' + By = D$ then $y' = f(t,y) = D - By$

So, $\frac{\partial f}{\partial y} = -B$. Since B, D are cont, our hypotheses are satisfied.

Ex: $y' = y^2$
$y(0) = 1$

Separable! $-\frac{1}{y} = t + c$

$\longrightarrow y = -\frac{1}{t+c}$; $y(0) = -\frac{1}{c} = 1$
$\longrightarrow c = -1.$

So $y(t) = -\frac{1}{t-1} = \frac{1}{1-t}.$

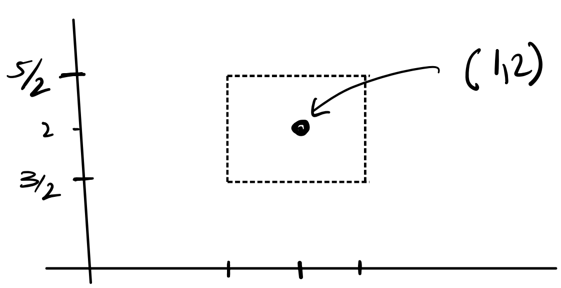
• $y'(t) = \frac{1}{(1-t)^2} = y^2$ and $y(0)=1$. So yes this is a sol^{n}. But... at $t=1$, $y=\infty$ so, the sol^{n} does not persist for all time! THIS obstacle what not obvious in the problem!

Ex: $\begin{cases} ty' + 2y = 4t^2 \\ y(1) = 2 \end{cases} \longrightarrow \begin{array}{l} y' = 4t - \frac{2}{t}y \\ y(1) = 2. \end{array}$

$f(t,y) = 4t - \frac{2}{t}y$; $f_y(t,y) = -\frac{2}{t}$

Both are continuous away from $t=0$. For example

$(1,2) \in R = \left(\frac{1}{2}, \frac{3}{2}\right) \times \left(\frac{3}{2}, \frac{5}{2}\right)$



So, by our theorem, as $f(t,y)$ and $f_y(t,y)$ are cont on R, there is a unique sol^n to the problem!

- $y' + \frac{2}{t}y = 4t$

$$\mu(t) = \exp(2\ln(t)) = \exp(\ln(t^2)) = t^2$$

$$y(t) = \frac{1}{t^2}\left[\int 4t^3\,dt + C\right]$$

$$= \frac{1}{t^2}\left[t^4 + c\right]$$

$$= t^2 + \frac{c}{t^2} \quad \text{for } y(1) = 1 + C = 2 \to C = 1$$

So $y(t) = t^2 + \frac{1}{t^2}$. * issue at $t=0$!

Solution Method 3: Exact Equations.

A special example: $\cdot\, y(1) = 0$

(*) $2x + y^2 + 2xyy' = 0.$

This is a non-linear separable eq^n.

But: If we set

$$\psi(x,y) = x^2 + xy^2$$

Then, $\psi_x = 2x + y^2$, $\psi_y = 2xy$. So, our eq^n

is $\psi_x + \psi_y y' = 0.$

But we can rewrite this as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial x} = 0$$

i.e. $\frac{d}{dx}\psi(x,y) = 0$

and so $\psi(x,y) = C$ implicitly defines solutions of (*). Since $y(1)=0$

we find $\psi(1,0) = 1$ giving $C = 1$.

Our solution is given by

$$x^2 + xy^2 = 1$$

$$\left(y = \sqrt{\frac{1-x^2}{x}} \right)$$

THIS IS a special example of an exact equation.

- Fix (x_0, y_0) and let M, N be continuous in a rectangle R containing (x_0, y_0). An equation

$$M(x,y) + N(x,y)\,y' = 0 \qquad (*)$$

is exact if

$$M_y = N_x \quad \text{in } R.$$

so there is a ψ s.t. $\psi_x = M$, $\psi_y = N$

- THE Solution of * is given implicitly by

$$\psi(x,y) = C$$

Ex: $(y\cos(x) + 2xe^y) + (\sin(x) + x^2 e^y - 1)y' = 0$

Find all sol^{n}'s:

$M = y\cos(x) + 2xe^y$; $M_y = \cos(x) + 2xe^y$

$N = \sin(x) + x^2 e^y - 1$; $N_x = \cos(x) + 2xe^y$

Since $\psi_x = M$,

$$\psi(x,y) = y\sin(x) + x^2 e^y + C(y)$$

$$\psi_y = \sin(x) + x^2 e^y + C'(y) = \sin(x) + x^2 e^y - 1 = N$$

So, $C'(y) = -1 \longrightarrow C(y) = -y$ and we find

$$\psi(x,y) = y\sin(x) + x^2 e^y - y$$

and solutions of our eq^{n} are given by

$$\psi(x,y) = C \text{ for a constant } C \in \mathbb{R}.$$

Ex: Every separable eq^{n} is exact.

- $y' = \psi(x)\varphi(y)$

So, $\dfrac{1}{\varphi(y)} y' - \psi(x) = 0$

i.e. set $M = -\psi(x)$, $N = \dfrac{1}{\varphi(y)}$.

Now $M_y = 0 = N_x$! it's exact!

Linear Equations are separable and so also exact.

Not every eq^n is exact. But, sometimes, we can find an integrating factor $\mu(x,y)$ s.t.

$M(x,y) + N(x,y)y' = 0$ INEXACT

$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)y' = 0$ EXACT.

i.e. $(\mu M)_y = (\mu N)_x$

i.e. $\mu_y M - \mu_x N = \mu(N_x - M_y)$

$\mu = \mu(x)$

$$\mu' = \mu\left(\frac{M_y - N_x}{N}\right)$$

- if $\frac{M_y - N_x}{N}$ is a function of x only, proceed.

$\mu = \mu(y)$

$$\mu' = \mu\left(\frac{N_x - M_y}{M}\right)$$

- if $\frac{N_x - M_y}{M}$ is a function of y only, proceed

THESE are Separable Equations and easy to solve!

<u>Ex:</u> $y + (2xy - e^{-2y})y' = 0$

$M = y \qquad M_y = 1$

$\neq$ INEXACT!

$N = 2xy - e^{-2y} \qquad N_x = 2y$

<u>But:</u>
- $\dfrac{M_y - N_x}{N} = \dfrac{1-2y}{2xy - e^{-2y}}$ depends on y!
- $\dfrac{N_x - M_y}{M} = \dfrac{2y-1}{y} = 2 - \dfrac{1}{y}$ depends only on y!

There is an integrating factor $\mu = \mu(y)$ satisfying

$$\mu' = \mu\left(2 - \frac{1}{y}\right)$$

i.e. $\ln \mu = 2y - \ln(y)$

so $\mu = e^{2y} \cdot e^{-\ln(y)} = \dfrac{1}{y} e^{2y}$

Multiplying our eq$^{\underline{n}}$ by μ,

$$e^{2y} + \left(2xe^{2y} - \frac{1}{y}\right)y' = 0$$

$\tilde{M} = e^{2y} \qquad \tilde{M}_y = 2e^{2y}$

$\tilde{N} = 2xe^{2y} - \dfrac{1}{y} \qquad \tilde{N}_x = 2e^{2y}$ ✓ it's exact.

- $\psi_x = e^{2y} \rightarrow \psi = xe^{2y} + C(y)$

$$\rightarrow \psi_y = 2xe^{2y} + C'(y)$$

$$= 2xe^{2y} - \frac{1}{y}$$

$$\rightarrow C'(y) = -\frac{1}{y} \rightarrow C(y) = \ln(y)$$

So $\psi(x,y) = xe^{2y} + \ln(y)$

and our solutions are given by

$xe^{2y} + \ln(y) = C$ for constants C.