Repeated Real Roots:

Ex: Find the sol^{n} of

$$\begin{cases} y'' + 4y'' + 4y = 0 \\ y(0) = 1,\ y'(0) = 2 \end{cases}$$

Sol^{n}: Our characteristic poly^{n} is $r^2 + 4r + 4 = (r+2)^2 = 0$

So, $r = -2$ is the only root. This gives 1 sol^{n}

$$y_1(t) = e^{-2t}$$

where can we find a second sol^{n}?

Notice: Set $y_2(t) = te^{-2t}$.

$$y_2'(t) = e^{-2t} - 2te^{-2t}$$

$$y_2''(t) = -2e^{-2t} - 2e^{-2t} + 4te^{-2t}$$
$$= -4e^{-2t} + 4te^{-2t}$$

THEN: $y_2'' + 4y_2' + 4y_2$

$$= \cancel{-4e^{-2t}} \cancel{+4te^{-2t}} + \cancel{4e^{-2t}} - \cancel{8te^{-2t}} + \cancel{4te^{-2t}}$$
$$= 0$$

$$y_1(t) = e^{-2t},\quad y_2(t) = te^{-2t}$$

So, $W(y_1, y_2) = \det \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix}$

$= e^{-4t} - 2te^{-4t} + 2te^{-4t}$

$= e^{-4t} \neq 0$ for any $t \in \mathbb{R}$.

So, y_1, y_2 form an FFS.

Our Sol^n is then

$y(t) = Ae^{-2t} + Bte^{-2t}$

$y(0) = A + B = 1 \longrightarrow B = 1 - A$

$y'(0) = -2A + B = 2 \longrightarrow B = 2 + 2A$

i.e. $1 - A = 2 + 2A$

$\longrightarrow 3A = -1$

$\longrightarrow A = -1/3. \longrightarrow B = 4/3$

So $y(t) = -\frac{1}{3}e^{-2t} + \frac{4}{3}te^{-2t}$.

<u>Theorem</u>: Let $a, b, c \in \mathbb{R}$ s.t. $ar^2 + br + c = 0$
$\Leftrightarrow (r-s)^2 = 0$.

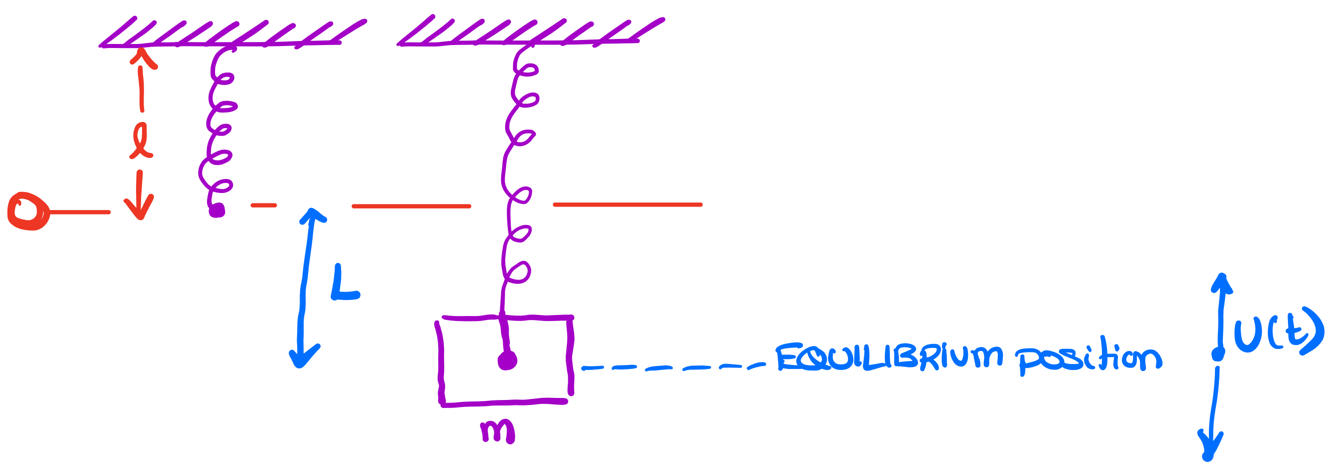
Then, the IVP

$$\begin{cases} ay'' + by' + cy = 0 \\ y(t_0) = y_0, \ y'(t_0) = y_1 \end{cases}$$

has sol^n

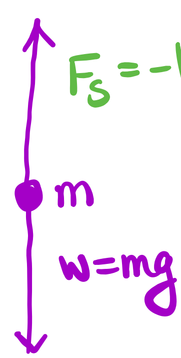
$y = Ae^{3t} + Bte^{3t}$ for constants A, B to be determined.

Example: a spring-mass system



* $U(t)$ measures displacement of m at time t from equilibrium.

THE Forces acting on our mass At Equilibrium.



THE spring restorative force.

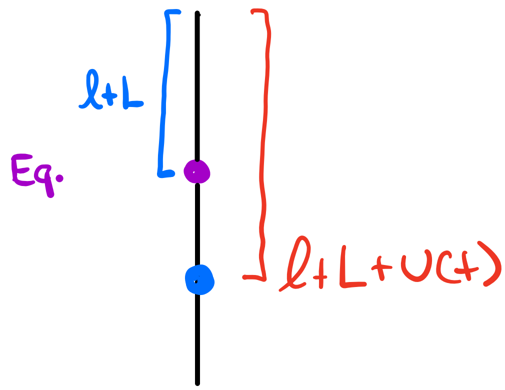
gravity pulls down and we will use downward as the positive displacement direction

Since there are no other forces, we have

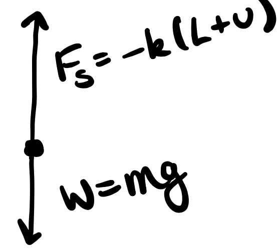
$$mg - kL = 0$$

$$\text{i.e. } k = \boxed{\frac{mg}{L}} \; ; \; g = 9.8\,m/s.$$

Forces acting in the non-equilibrium state.



Forces



depends only on displacement from spring length ℓ!

Damping: $F_d = -\gamma u'(t)$ acting in the opposite direction of velocity.

External Force: $F(t)$ acts only downward or upward.

Newton's Law: $F = ma$

i.e. $mu'' = mg - k(L+u) - \gamma u' + F(t)$

$$= \overbrace{mg - kL}^{=0} - ku - \gamma u' + F$$

i.e. $mu'' + \gamma u' + ku = F$. So, $u(t)$ must satisfy this second order problem.

if the external force $F(t) \neq 0$, this is an in-homogeneous eqⁿ. More on this later!

- Suppose a 10 kg weight hangs from a spring with length 5m. When the weight is attached, the spring extends to 5.5m. Assuming zero friction and that the mass begins at $u(0) = 1$m, find $u(t)$, the function governing the displacement of the mass part eq^n.

$$k = \frac{mg}{L} = \frac{10 \cdot 9.8}{1/2} = \frac{98}{2} = 49.$$

The equation satisfied by $u(t)$ is

$$10u'' + 49u = 0$$

$$\longrightarrow \quad u'' + \frac{49u}{10} = 0 \qquad \bullet \; r^2 + \frac{49}{10} = 0$$

$$\text{i.e. } r = \pm \frac{7}{\sqrt{10}} i$$

$$\text{i.e. } y(t) = Ae^{i\frac{7}{\sqrt{10}}t} + Be^{-\frac{7}{\sqrt{10}}ti}$$

$$= \bar{A}\sin\left(\frac{7}{\sqrt{10}}t\right) + \bar{B}\cos\left(\frac{7}{\sqrt{10}}t\right)$$

$$y(0) = 1 = \bar{B}$$

$$y'(0) = \frac{7}{\sqrt{10}}\bar{A} = 0.$$

So, $y(t) = \cos\left(\frac{7}{\sqrt{10}}t\right)$

Suppose now, the spring is damped with constant $\gamma = -1/2$.

Then, $10u'' + \frac{1}{2}u' + 49u = 0$, $u(0) = 1$, $u'(0) = 0$.

$$10r^2 + \frac{1}{2}r + 49 = 0$$

$$\text{i.e. } r^2 + \frac{1}{20}r + \frac{49}{10} = 0$$

$$\text{i.e. } r = -\frac{1}{40} \pm \frac{1}{2}\cdot\sqrt{\frac{1}{400} - \frac{4 \cdot 49}{10}}$$

$$\frac{-1959}{400}$$

$$= -\frac{1}{40} \pm \frac{1}{40}\sqrt{1959}\, i$$

So $u(t) = e^{-\frac{1}{40}t}\left[A\sin\left(\frac{\sqrt{1959}}{40}t\right) + B\cos\left(\frac{\sqrt{1959}}{40}t\right)\right]$

$u(0) = B = 1$

$u'(0) = \frac{\sqrt{1959}}{10}A = 0 \longrightarrow A = 0$

So $u(t) = e^{-\frac{1}{40}t}\cos\left(\frac{\sqrt{1959}}{40}t\right)$

See animations