2208 Lecture 6

Recall, so far, given 2 solutions y_1, y_2 of

$$① \quad y'' + p(t)y' + q(t)y = 0$$

and we also know $y(t) = Ay_1(t) + By_2(t)$

is also a sol^{n}.

To find A, B we make use of initial conditions

- $y(t_0) = y_1, \; y'(t_0) = y_2.$

Is it always possible to find $A, B \in \mathbb{R}$ s.t. y

satisfies ① and ②?

$$\boxed{\begin{aligned} \cdot\; y(t_0) &= Ay_1(t_0) + By_2(t_0) = y_0 \\ \cdot\; y'(t_0) &= Ay_1'(t_0) + By_2'(t_0) = y_1 \end{aligned}}$$

So,

$$A = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_1 & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad B = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_1 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

and these are real #'s provided

$$W(y_1,y_2) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)\,y_2'(t_0) - y_1'(t_0)\,y_2'(t_0) \neq 0.$$

If $W(y_1,y_2)=0$ at t_0, the equation may have no solution at all!

THEOREM: Suppose y_1, y_2 satisfy

$$(*)\quad y'' + p(t)y' + q(t)y = 0.$$

Then, given any t_0, we may find a sol$^{\underline{n}}$ of $(*)$ in the form $y = Ay_1 + By_2$ iff $W(y_1,y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \neq 0.$

Ex: From last time, we found

$$y'' + 3y' + 2y = 0, \quad t \in \mathbb{R}.$$

has solutions $y_1 = e^{-t}$, $y_2 = e^{-2t}$. Notice,

$$W(y_1,y_2) = \det\begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix}$$

$$= -2e^{-3t} + e^{-3t}$$

$$= -e^{-3t} \neq 0 \text{ for any } t \in \mathbb{R}.$$

So, no matter t_0, y_0, there is a sol^n

$$y = Ae^{-t} + Be^{-2t} \text{ satisfying } y(t_0) = y_0, y'(t_0) = y_1.$$

Def^n: Given $(*)$ $y'' + p(t)y' + q(t)y = 0$ with p, q continuous on $I = (a,b)$, we say two solutions y_1, y_2 form a Fundamental Sol^n set if

$$W(y_1, y_2) \neq 0 \text{ for all } t \in I.$$

Theorem: Given the same eq^n, let y_1 satisfy

$$y'' + p(t)y' + y = 0$$
$$y(t_0) = 1, \; y'(t_0) = 0$$

and y_2 satisfy

$$y'' + p(t)y' + q(t)y = 0$$
$$y(t_0) = 0, \; y'(t_0) = 1.$$

Then, y_1 & y_2 form a FSS for $(*)$.

Ex: $y'' - y = 0$; $t_0 = 0$. Find an FSS.

Sol^n: Ch. Eq^n is $r^2 = 1 \to r = \pm 1$. So e^{-t} and e^t and the most general form of a sol^n is

$$y = Ae^{-t} + Be^{t}.$$

Initial Conditions:

$y(0) = 1,\ y'(0) = 0 \longrightarrow \begin{array}{l} A + B = 1 \\ -A + B = 0 \end{array}$ $\quad B = 1/2,\ A = 1/2.$

$y(0) = 0,\ y'(0) = 1 \longrightarrow \begin{array}{l} A + B = 0 \\ -A + B = 1 \end{array}$ $\quad B = 1/2,\ A = -1/2.$

Set $y_1(t) = \frac{1}{2}\left(e^{-t} + e^{t}\right) = \cosh(t)$

$$y_2(t) = -\frac{1}{2}\left(e^{-t} - e^{-t}\right) = \sinh(t)$$

$$\cdot\ W\big(\cosh(t), \sinh(t)\big) = \det\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix}$$

$$= \cosh^2(t) - \sinh^2(t)$$

$$= 1 \neq 0.$$

So, every sol^{n} of $y'' - y = 0$ may be written as

$$y(t) = c_1 \cosh(t) + c_2 \sinh(t)$$

Complex Roots of the Ch. Poly^{n}:

Euler's Formula:

For any $\theta \in \mathbb{R}$,

$$\boxed{i^2 = -1}$$

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

- $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

To see why, recall, for any t,

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

So, $e^{it} = \displaystyle\sum_{n=0}^{\infty} \frac{(it)^n}{n!}$

$$= 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

$$= \cos(t) + i\sin(t).$$ Wow!

Ex: Solve the IVP:

$$y'' + y = 0$$
$$y(0) = 1,\ y'(0) = 0$$

6

Sol^n: The ch. poly^n is $r^2+1=0$ with solutions $r_1=i$, $r_2=-i$.

Our solutions are

$$y_1(t)=e^{it}$$
$$y_2(t)=e^{-it} \longrightarrow y(t)=Ae^{it}+Be^{-it}$$

Do y_1, y_2 form an FSS?

$$W(e^{it}, e^{-it}) = \det\begin{pmatrix} e^{it} & e^{-it} \\ ie^{it} & -ie^{-it}\end{pmatrix}$$

$$= -i-i = -2i \neq 0$$

yes!

$$y(0) = A+B = 1 \longrightarrow A = 1-B$$

$$y'(0) = iA - iB$$
$$= i(A-B) = 0 \longrightarrow A = B \longrightarrow B = 1-B \longrightarrow B = 1/2 \longrightarrow A = 1/2$$

So, $y(t) = \frac{1}{2}(e^{it}+e^{-it})$

$= \cos(t)$. ☺

7

THM: Let $a, b, c \in \mathbb{R}$ s.t. $ar^2+br+c=0$ has 2 complex roots

$$r_1 = a+ib$$
$$r_2 = a-ib.$$

Then, the solution of the IVP

$$\begin{cases} ay''+by'+cy=0 \\ y(t_0)=y_0 \\ y'(t_0)=y_1 \end{cases}$$

has the unique sol^{n}:

$$* \; y(t) = e^{at}\left(A\cos(bt) + B\sin(bt)\right)$$

where A, B are found through linear algebra.

To see why, we just re-express $e^{(a+ib)t}$

- $e^{(a+ib)t} = e^{at} \cdot e^{ibt}$

$$= e^{at}\left(\cos(bt) + i\sin(bt)\right)$$

- $e^{(a-ib)t} = e^{at}\left(\cos(bt) - i\sin(bt)\right)$

So, $Ae^{(a+ib)t} + Be^{(a-ib)t} = e^{at}\left(A\cos(bt) + B\sin(bt)\right)$

* Notice, $y_1(t) = e^{at}\cos(bt)$, $y_2(t) = e^{at}\sin(bt)$

$$\det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = e^{at}\det\begin{bmatrix} \cos bt & \sin bt \\ -b\sin bt & b\cos bt \end{bmatrix}$$

$$= e^{at}\left[b\cos^2 bt + b\sin^2 bt\right]$$

$$= be^{at}.$$

Since $b \neq 0$, $W(y_1, y_2) \neq 0$ for any $t, a \in \mathbb{R}$.
i.e. y_1, y_2 form an FFS for the problem.