We have seen methods for solving

- Linear: $y' + B(t)y = D(t)$, $y(t_0) = y_0$
- Separable: $y' = \varphi(t)\psi(y)$, $y(t_0) = y_0$

and
- Exact Equations: $M(x,y) + N(x,y)y' = 0$ where $M_y = N_x$.

Lemma: Every linear equation with continuous coefficients has an exact integrating factor.

Proof: For functions $B(x), D(x)$ consider

$$y' + B(x)y = D(x)$$

Multiplying by $\mu = \exp \int B(x)dt$ gives

$$\frac{d}{dx}(\mu y) = \mu D$$

Set $w = \mu y$ and we find the new eq^n

$$w' = \mu D.$$

This is in the form

$$M(x,w) + N(x,w)w' = 0$$

where $M = -\mu D$, $N = 1$.

$$M_y = 0 = N_x$$ so it is exact!

- $\Psi_w = N \longrightarrow \Psi(x,w) = w + C(x)$

$$\Psi_x = C'(x) = -\mu D$$

So, $C(x) = -\int_{t_0}^{t} \mu D \, ds + C_1$ for some $C \in \mathbb{R}$.

All solutions are then given implicitly by

$$w - \int_{t_0}^{t} \mu(s) D(s) ds = C \quad \text{for } C \in \mathbb{R}.$$

i.e. $y = \frac{1}{\mu}\left[\int_{t_0}^{t} \mu(s) D(s) ds + C\right]$

the same solution!!

Lemma: Every separable equation is exact.

$$y' = \varphi(t)\psi(y)$$

$$\longrightarrow \quad -\varphi(t) + \frac{1}{\psi(y)} y' = 0$$

This equation is $M(t,y) + N(t,y) y' = 0$

with $M = -\varphi(t)$, $N = \frac{1}{\psi(y)}$.

So, $M_y = 0 = N_t$ and we find all eq^n

to be exact. //

Second Order Linear Equations

General Form: The second order linear IVP:

$$(*) \begin{cases} y'' + p(t)y' + q(t)y = g(t) & \text{for } t \in (a,b) \\ y(t_0) = y_0, \; y'(t_0) = y_0' & \text{for some } t_0 \in (a,b) \end{cases}$$

there are two initial conditions required for second order eq^n's.

Do Solutions Exist, are they Unique?

Thm: (3.2.1) If p, q, g are continuous functions on (a,b) then, there is a unique sol^n to $(*)$ valid for all time in (a,b).

Ex: Find the longest time interval on which a sol^n of

$$\begin{cases} (1-t)^2 y'' + 2t y' - \frac{1}{t} y = 0 \\ y(3/4) = 2, \; y'(3/4) = 3. \end{cases}$$

exists.

Sol^n: Our eq^n is

$$y'' + \frac{2t}{(1-t)^2} y' - \frac{1}{t(t-1)} y = 0$$

Here $p(t) = \frac{2t}{1-t^2}$, $q(t) = \frac{1}{t(t-1)}$

and $g(t) = 0$.

Since p & q are both cont. on $(0,1)$ with discontinuity for q at $t=0$ and $t=1$ and p discont. at $t=1$, the longest interval we can be guaranteed a solution is, by our theorem, $t \in (0,1)$. □

Ex: Consider $\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = 0, \; y'(t_0) = 0. \end{cases}$ with p, q cont. in (a,b).

Clearly $y(t) = 0$ is a sol^n of this problem. By our theorem, p, q are cont with $g(t) = 0$, and so the sol^n of our problem is unique.

Thus, $y(t) = 0$ is the ONLY sol^n of this problem.

Resolution of the eq$^{\underline{n}}$ with const. coefficients.

- $ay'' + by' + cy = 0$.

Euler: Suppose $y = e^{rt}$ for some $r \in \mathbb{R}$. Then,

$$y = e^{rt},\ y' = re^{rt},\ y'' = r^2 e^{rt}$$

$$ay'' + by' + cy = (ar^2 + br + c)e^{rt}$$

Since $e^{rt} \neq 0$, our eq^{n} is satisfied only if

$$\boxed{\cdot\ ar^2 + br + c = 0}\ (*)$$

THIS IS called the Characteristic Eq$^{\underline{n}}$ Associated to • THERE ARE 3 Different Cases to consider.

- Distinct Real Roots.

THE sol$^{\underline{n}}$'s of $ar^2 + br + c = 0$ are $r = r_1, r_2 \in \mathbb{R}$.

Each of $y_1(t) = e^{r_1 t}$, $y_2(t) = e^{r_2 t}$ satisfy "•". By linearity, $y = Ay_1 + By_2$ for constants A, B is the most general sol$^{\underline{n}}$.

Ex: Solve the IVP

$$y'' - y = 0,\quad y(0) = 1,\ y(1) = 2$$

Sol$^{\underline{n}}$: Our Characteristic eq$^{\underline{n}}$ is

$$r^2 - r = 0 \longrightarrow r = 0, 1$$

are our solutions.

So, set $y = Ae^{0\cdot t} + Be^t$

$= A + Be^t$. ; $y' = Be^t$

$y(0) = A + B = 1$

$y'(1) = B \cdot e = 2 \longrightarrow B = \frac{2}{e}$.

Since $A = 1 - B = \frac{e-2}{e}$, we find

$$y(t) = \frac{e-2}{e} + \frac{2}{e}e^t$$

$$= \frac{1}{e}\left(e - 2 + 2e^t\right)$$

<u>Ex:</u> <u>Solve:</u> $y'' + 3y' + 2y = 0$

$y(0) = 1 = y'(0)$

<u>Sol^n:</u> $r^2 + 3r + 2 = (r+1)(r+2)$

So, our most general sol^n is

$$y(t) = Ae^{-t} + Be^{-2t}$$

for constants A, B.

Since $y(0) = A + B$, $y'(0) = -A - 2B$

we see $B = 1 - A$ giving $-A - 2(1-A) = 1$

i.e. $A - 2 = 1 \longrightarrow A = 3, \ B = -2.$

So our sol^n is

$$y(t) = e^{-t} - 2e^{-2t}.$$