Math 2208 Lecture 1

You have already seen differential equations in action!

i) Newton's law of cooling

$$u'(t) = k(u(t) - T_0) \quad (*)$$

- $u(t)$ is temperature of the object at time t
- T_0 is the temp. of the surrounding medium.

The Solution of $(*)$ is a function $u(t)$ that satisfies $(*)$. More, it gives the temp. of an object at any time t.

① Consider the eq^n

$$u'(t) = ku(t).$$

- Notice that $u(t) = e^{kt}$ solves this eq^n.

Suppose now there is another sol^n, say $w(t)$.

$$\left(\text{i.e. } w \text{ satisfies } w' = kw \right)$$

Then,
$$\frac{d}{dt}\left(\frac{w(t)}{e^{kt}}\right) = \frac{w'(t)e^{kt} - kw(t)e^{kt}}{e^{2kt}}$$
$$= \frac{(w'(t) - kw(t))}{e^{kt}}$$

$= 0$!

That is, $\frac{w(t)}{e^{kt}}$ is constant, i.e. there is a $C \in \mathbb{R}$ s.t. $w(t) = Ce^{kt}$.

②

② Set $g(t) = U - T_0$. Then $g'(t) = U'(t)$ and

$$g'(t) = k(U - T_0) = kg(t).$$

Thus, by part ①, $g(t) = Ce^{kt}$ for some $C \in \mathbb{R}$ and we find

$$U(t) = g(t) + T_0 = Ce^{kt} + T_0.$$

* this means that every solution of

$$U'(t) = k(U(t) - T_0)$$

is of the form

$$U(t) = Ce^{kt} + T_0$$

③ THERE are ∞-ly many solutions to our problem.

i.e. $U(t) = Ce^{kt} + T_0$ for any $C \in \mathbb{R}$.

To find a single solution, we need more info

"Initial Conditions"

Ex: • $U(0) = 17$ (we know temp at $t = 0$) $U(t_0) = S$
• $U'(0) = 6$ (we know rate of ch. of temp)

$u'(t_0) = S$

Using ②, $u(0) = C + T_0 = 17$ giving $u(t) = (17 - T_0)e^{kt} + T_0$.

③

The above involved the example of a 1^{st} order, linear diff^{n} eq^{n}: $u' - ku + kT_0 = 0$. In physical theories and in engineering applications, we can come into contact with a myriad of different first order eq^{n}'s and so, let's approach them generally:

- A first order eq^{n} is of the form

$$(*) \quad y' = F(t, y) \; ; \; y = y(t)$$

where $F : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is any function.
Here, we understand that $y = y(t)$, $y' = y'(t)$.

- Linear vs. Non-linear Equations.

$(*)$ is linear if for any $\alpha, \beta, w, y \in \mathbb{R}$

$$F(t, \alpha w + \beta y) - \alpha F(t, w) - \beta F(t, y)$$

is a function of t only.

Example: Newton's Law of Cooling:

$$u' = k(u - T_0) : \; F(t, u) = ku - kT_0.$$

Now,

$$F(t, \alpha w + \beta y) - \alpha F(t, w) - \beta F(t, y)$$

$= k(\cancel{\alpha w}+\cancel{\beta y}) - kT_0 - \alpha k \cancel{w} + \alpha k T_0 - \beta k \cancel{y} + \beta k T_0$

$= k(\alpha+\beta-1)T_0 \longleftarrow$ independent of w, y!

In Fact, the general first order linear equation is always of the form

$$(**) \qquad y' + B(x)y + C(x) = 0$$

for some functions B, C, $x \in [a,b]$

Solving The General Linear Equation:

Set $\mu(x) = \int_a^x B(s)ds$. Then, $\frac{d}{dx} e^{\mu(x)} = B(x)e^{\mu(x)}$

and so, our equation is equivalent to

$$\frac{d}{dx}\left[\mu(x)y\right] = -\mu(x)C(x).$$

Thus,

$$(*) \; y(x) = \frac{-1}{\mu(x)}\left[\int_a^x \mu(s)C(s)ds \, . + C\right]$$

is a solution to the equation.

THEOREM: Given any linear, first order eq^{n}.

$y' + B(x)y + C(x) = 0$ for functions continuous on $[a,b]$, the solution satisfying $y(a) = C$ is $(*)$.

THEOREM: (UNIQUENESS) Under the hypotheses of the previous theorem, the solution $y(x)$ is the only solution satisfying $y(a) = C$.

If there are two solutions y, w then $v = y - w$ satisfies $v' + B(x)v = 0$ and $v(a) = y(a) - w(a) = C - C = 0$.
More, $\frac{d}{dx}\left(\mu(x)v\right) = 0$ giving $v = \frac{k}{\mu(x)}$. Since $v(a) = 0 \Rightarrow k = 0$ and so v is constant. Since $v(a) = 0$, $v = 0$ and we see $y(x) = w(x)$. □

5

THE solution $y = \frac{-1}{\mu}\left(\int \mu c\, dx + C_1\right)$ is a 1-parameter family of solutions to our problem. THE parameter indexing our family is THE constant of integration. This family forms the GENERAL SOLUTION of our equation.

Given more information, the constant C can be determined.

Examples:

- NEWTON'S LAW of Cooling: For some given $k \neq 0$, $T_0 \in \mathbb{R}$,

Find the general solution of the Newton model

$$T' = k(T - T_0) \quad ; \quad t \in [0, \infty)$$

Rewrite

$$T' - kT + kT_0 = 0 \quad \text{it's } 1^{ST} \text{order, linear.}$$

$$\rightarrow T(t) = -\frac{1}{\mu(t)} \int_0^t \mu(s) \cdot kT_0\, ds$$

where $\mu(t) = \exp\left[\int_0^t (-k)\, ds\right] = \exp\left[-kt\right] = e^{-kt}$

giving

$$T(t) = -e^{kt}\int e^{-ks}\cdot kT_0\,ds$$

$$= -kT_0 e^{kt}\left[-\frac{1}{k}e^{-kt} + C\right]$$

$$= T_0 - Me^{kt}.$$

Does it work? $T' = -kMe^{kt}$

So $T' - kT + kT_0 = -k\cancel{Me^{kt}} - k(\cancel{T_0} - \cancel{Me^{kt}}) + \cancel{kT_0}$
$= 0$!

It seems that linear equations are rather straight forward.

<u>Non-Linear Equations:</u>

- $y'\sin(y) = 3ye^{y'}$
- $yy' = 3$ and many more. The resolution of these

equations depends on our point of view.

(I) Separable Equations.

A First order equation is called Sep. if there are functions $f(x)$, $g(y)$ so that our eqn is equivalent to

$$\frac{d}{dx}\left(g(y)\right) = f(x)$$

$\longrightarrow$ it's easy to see $y(x) = g^{-1}\left(\int f(x)\,dx\right)$

is the solⁿ of the problem.

Ex: $\ln(y') = y+1$.

then $y' = e^{y+1} \longrightarrow e^{-y} y' = e$

$$\longrightarrow \int e^{-y}\,dy = \int e \cdot dx$$

$$\rightarrow -e^{-y} = ex + C$$

$$e^{-y} = C - ex$$

$$y = -\ln(C-ex)$$

$$= \ln \frac{1}{C-ex}$$ for any

constant C.

Check: $y' = -\frac{-e}{C-ex} = \frac{e}{C-ex}$

$\ln(y') = \ln(e) - \ln(C-ex)$ $\qquad y+1 = 1-\ln(c-ex)$ ✓

$= 1 - \ln(C-ex)$

Ex: $(y')^2 - 36xy = 0$; $y \geq 0$.

(*) $y' = 6\sqrt{x}\sqrt{y}$

So $\int \frac{dy}{y^{1/2}} = \int 6x^{1/2}dx \longrightarrow 2y^{1/2} = \frac{12}{3}x^{3/2} + C$

$$= 4x^{3/2} + C.$$

So $y(x) = \left[2x^{3/2} + c\right]^2$ for some $C \in \mathbb{R}$.

Check: $y' = 2[2x^{3/2}+c]\cdot 3x^{1/2} = 6\sqrt{x}(2x^{3/2}+c)$ ✓

$6\sqrt{x}\sqrt{y} = 6\sqrt{x}\cdot(2x^{3/2}+c)$ ✓

LECTURE II :

1

① Equation : 1ST order.

$$F(t, y, y') = 0$$

THIS IS your model develloped by physical laws

② General Solⁿ is a 1-parameter family of functions $y(t)$ all of whom satisfy our equation.

THE Solution governs the behaviour of the quantity you measure!

③ Particular Solution. A function $y(t)$ chosen from the 1-parameter family that satisfies a condition.

Ex: $y(t_0) = Y_0$

THESE ARE Called initial conditions.

Ex: A bacterium grows in a dish with enough food supply to support at most 1000 cells. If there are 100 cells in the dish at the outset at the number of cells grows in proportion to the Carrying capacity less the popⁿ. Find the popⁿ at t hrs.

- $P(t)$ is popⁿ at time t.
- $P'(t) = kP(1000-P)$ THIS IS a sep. eqⁿ!

$$\frac{dP}{P(1000-P)} = k\,dt$$

$$=$$

$$\frac{1}{1000}\left(\frac{1}{1000-P} + \frac{1}{P}\right)dP = k\,dt$$

so $\frac{1}{1000}\Big(\ln P - \ln(1000-P)\Big) = kt + C$

i.e. $\ln \frac{P}{1000-P} = 1000kt + C$

$$\frac{P}{1000-P} = Ae^{1000kt}$$

$\rightarrow P(t) = \frac{1000Ae^{1000kt}}{1+Ae^{1000kt}}$; $P(0) = \frac{1000A}{1+A} = 100$

$$\rightarrow \frac{A}{1+A} = 1/10$$

$$A(1-1/10) = 1/10$$

$$A = \frac{1/10}{9/10} = 1/9$$

$P(t) = \frac{9000e^{1000kt}}{1+1/9\,e^{1000kt}}$ Plot graph

Some dynamics: Here, we consider the special

1^{st} order eqⁿ $\begin{cases} y' = f(x,y) \\ y(x_0) = Y_0. \end{cases}$

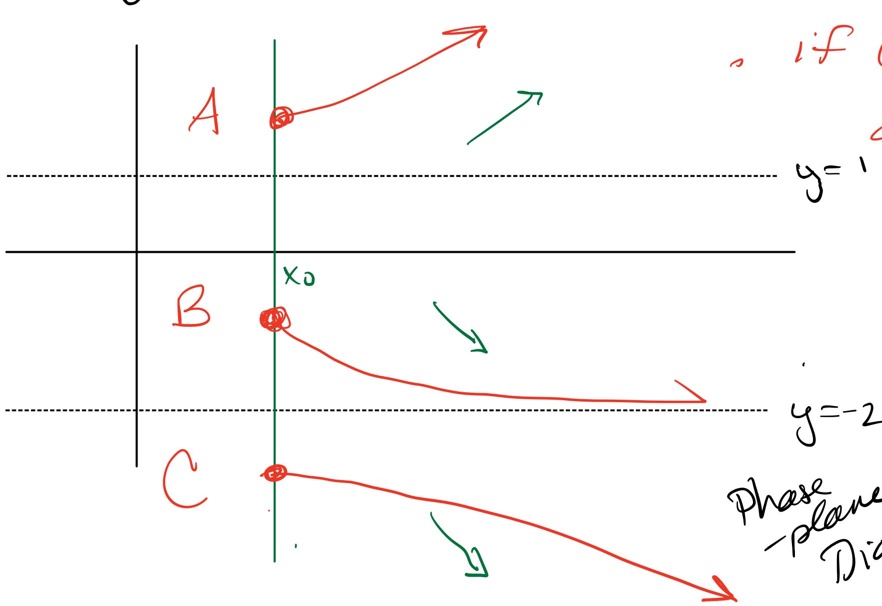
A solution to this problem is a diffⁿ function $y(x)$ satisfying $y' = f(x,y)$ and passing through the point (x_0, Y_0). THE Equation tells us a lot!

- if $f(x,y) \geq 0$ then $y' \geq 0$ and so $y \nearrow$
- if $f(x,y) \leq 0$ then $y' \leq 0$ and $y \searrow$

Ex: $y' = (y-1)(y+2)^2$; $y(x_0) = y_0$

- $f(x,y) = (y-1)(y+2)^2 = 0$ if $y = 1$ or $y = -2$.
- $y \in (-\infty, -2)$: $f < 0$ $\searrow$
- $y \in (-2, 1)$: $f < 0$ $\searrow$
- $y \in (1, \infty)$: $f > 0$ $\nearrow$

- if $(x_0, y_0) \in A$, $y(x) \to \infty$ as $x \to \infty$.
- if $(x_0, y_0) \in B$, $y(x) \to -2$ as $x \to \infty$
- if $(x_0, y_0) \in C$ then $y(x) \to -\infty$



Phase-plane Diagram.

The lines $y=1$, $y=-2$ are called equilibrium lines.

Let's talk in more generality:

- $y' = f(x,y)$; Equilibria are given by $f(x,y)=0$.
- An Equilibrium is STABLE if all nearby trajectories converge to it
- UNSTABLE if no nearby trajectories converge to it
- Semi-Stable obvious def^n. (our last ex has 1 unstable and 1 semi-stable eq^m)

Ex: Consider the eq^n,

$$y' = y(y-1)^2(y-2)$$

and suppose $y(0) = 3/2$. Find $\lim_{t\to\infty} y(t)$.

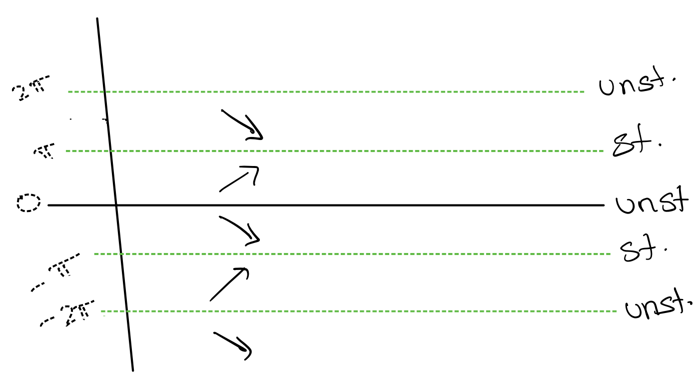
Sol^n: $y=0,1,2$ are equilibria.

$y'<0$ if $y\in(1,2)$ $\quad$ $3/2 \in (1,2)$ so $\lim_{t\to\infty} y(t) = 1$

$y'>0$ if $y\in(2,\infty)$

Ex: $y' = \sin(y)$

Equilibria at $y = 0, k\pi$; $k\in\mathbb{Z}$



- if $y(1) = 3\pi/2$, $y(t)\to\pi$
- if $y(1) = -\pi/2$, $y(t)\to -\pi$.