Math 422 2011 Assignment 3

March 14^{th}, 2011

Answer all questions and give complete reasons and checks for your answers. Please do not erase anything, just put a line through your work and continue; you cannot lose marks for anything you write. The parts of the questions are weighted as shown and can be answered in any order.

1. (a) Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean Domain using the standard norm.

   (b) Verify that $\mathbb{Z}[\sqrt{-5}]$ is not a PID by considering $J :=< 3, 1 + 2\sqrt{-5} >$.

   (c) Pick a prime $p > 5$ different from all others in the class and find elements in $\mathbb{Z}[\sqrt{-p}]$ to show it is not a Unique Factorisation Domain. [6]

2. We say that H and K are relatively prime ideals of a ring R if $H + K = R$. [9]

   (a) Explain why $H \cap K = HK$ and give an example (with neither ideal being the zero ideal) of when $H \cap K = \{0\}$.

   (b) Explain how this is related to the Chinese Remainder Theorem for integers and find a unique pair amongst your classmates of relatively prime ideals in $\mathbb{Z}[i]$.

   (c) Use the first isomorphism theorem and the obvious map (for any H and K, not necessarily your chosen ideals) from R to $(R/H) \times (R/K)$ to prove the Chinese Remainder Theorem for two ideals, ensuring your map is onto and a homomorphism.

3. (a) i. Find all monic (largest term x^2) irreducible polynomials of degree 2 in $\mathbb{Z}_5[x]$ and one of degree 3 (which is different from everyone else's in the class).

   ii. Give an example of a monic polynomial of degree 4 which is not irreducible but which doesn't evaluate to zero for $x = 0, \ldots, 4$.

   iii. Explain why it is sufficient to only characterise monic polynomials.

   iv. What percentage of polynomials of degree 2 and 3 are irreducible? [7]

   (b) Create the factor ring with respect to the principal ideal of your polynomial of degree 3. Identify what extension ring it is isomorphic to, showing that the isomorphism holds. [3]