Math422 Ring Theory: Assignment 1, January 2011

Please show all working and reasoning to get full marks for any question. Please only discuss general ideas with your classmates, all work handed in should be your own and plagiarism will be punished. Attach all rough work attempted to show your thought processes.

1. An element e in a ring R is called *nilpotent* if $e^k = 0$ for some $k \geq 1$.

   (a) Give examples of non-zero nilpotent elements in these rings or explain under what circumstances no non-trivial nilpotent element can exist. [6]

   - $\mathbb{Z}_m$ where m is a composite integer. Examine $\mathbb{Z}_{26}$ and $\mathbb{Z}_{56}$ to start with and try to make your answer as general as possible.
   - $n \times n$ matrices with entries from $\mathbb{Z}_p$ (where p is a prime integer and $n \geq 2$).

   (b) If e is nilpotent in R, prove that $(1 - e)$ must be a unit in R by finding an $f \in R$ such that $(1 - e)f = 1$. [3]

2. (a) Prove that every finite integral domain R is a field by considering the set aR for some non-zero $a \in R$. [5]

   (b) Choose a finite commutative product or quadratic extension ring T (different from everyone else in the class and containing between 17 and 40 elements) which isn't a field, explaining why it isn't a field. [1]

   (c) Working in your ring T, categorise which elements are units, which are zero divisors and which are prime or irreducible. What is the characteristic of T? [7]

3. Prove that the only ring which has $1 = 0$ is the trivial ring $\{0\}$ and give an example of a finite ring without unity. [3]