University College of Cape Breton

Math421

Modern Algebra I

November 2003

Time : 72 hours

Answer all questions, giving all working and reasoning.

Q1. The presentation for D_n is $\langle a, b : a^n = b^2 = e, aba = b \rangle$.

(a) Using a Cayley diagram, or otherwise, list all the elements of D_6 in terms of a and b and give the cyclic subgroups and hence the orders of each element.

(b) Carefully find all the subgroups of D_6, giving reasons why you have found all of them. Which are cyclic subgroups and which are normal subgroups?

(c) What is the centre of D_6, the centraliser of $a^2 b$ and the normaliser of $< ab >$?

(d) Using least common multiples, calculate the order of each element of $D_3 \times \mathbb{Z}_2$ and hence give and verify an isomorphic mapping between it and D_6.

(e) Prove that if p is prime then $D_{2p} \cong D_p \times \mathbb{Z}_2$, but $D_{4p} \ncong D_{2p} \times \mathbb{Z}_2$.

Q2. For each of these mappings from S_n, identify what the image set and the kernel is in each case.

(a) $\phi(p) = p^2$

(b) $\phi(p) = p^{-1}$

(c) $\phi(p) = (1\ 2)p$

(d) $\phi(p) = (1\ 2)p(1\ 2)$

(e) $\phi(p) =$ the length of the longest cycle in p

(f) $\phi(p) = 0$ if p is an even permuation, 1 otherwise

Which of the above are homomorphisms? Are any isomorphims?

Q3. (a) Prove that $H_1 := \{(0,0),(0,2),(2,0),(2,2)\}$ and $H_2 := \langle(2,3)\rangle$ are subgroups of $G := \mathbb{Z}_4 \times \mathbb{Z}_4$.

(b) What are G/H_1 and G/H_2? What groups are they isomorphic to?

(c) Show that $H_1 \cap H_2$ is a subgroup and find a proper subgroup of G which contains $H_1 \cup H_2$.

(d) Is it true that for any subgroups H_i and H_j of G that $H_i \cup H_j$ is always contained in a proper subgroup of G?

(e) Choose an element of G which, together with (2,3), generates G, and explain why this element can't come from H_1. Use these 2 elements to draw a Cayley diagram of G. Clearly identify the cosets of H_1 and H_2 on the diagram.

Q4. Define the set $S := \{p + qi : p, q \in \mathbb{Q}, i = \sqrt{-1}\}$, and recall the set R^* is the group of all elements in R apart from zero, with group operation multiplication.

(a) Use the subgroup test to prove that $\mathbb{Q}^* \leq S^* \leq \mathbb{C}^*$.

(b) Use the normal subgroup test to prove that $\mathbb{Q}^* \triangleleft S^* \triangleleft \mathbb{C}^*$.

(c) Describe the cosets of $\mathbb{Q}^*$ in S^*.

(d) Letting $W := S^*/\mathbb{Q}^*$, find three elements of W with order less than 10, identify W's group operation and check the four group axioms for W under this operation.

END OF QUESTION PAPER