Math 4204 Assignment 3: U.S.D.s/Eisenstein

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

1. A *universal side divisor* d in a ring A is an element which is not a unit but for every $a \in A$ either $d|a$ or there exists a unit $u \in A$ such that $d|(a-u)$.
   (a) Prove that if e is associate with a universal side divisor d then it too is one. Explain why the only universal side divisors in $\mathbb{Z}$ are ± 2 and ± 3 and find the universal side divisors in $\mathbb{Z}[i]$ and $\mathbb{Z}_q[x]$ for any prime q. [6]
   (b) Prove that if d is a universal side divisor then it is irreducible in A by supposing $d = bc$ and then concluding that either b or c is a unit. [2]

2. Recall that the Eisenstein integers $\mathbb{Z}[\omega]$ are the set of numbers of the form $s + t\omega$ where s and t are integers and $\omega = \frac{-1+\sqrt{-3}}{2}$. Answer these questions: [7]
   (a) i. Identify the associates of 2 and 3 and explain which of these twelve Eisenstein integers are prime in $\mathbb{Z}[\omega]$ and which are composite.
      ii. Show that 7 and 13 can be written as $N(s + t\omega) = s^2 - st + t^2$ for some integers s and t and deduce that they are not prime in $\mathbb{Z}[\omega]$.
      iii. Prove that $N(s+t\omega)$ is never congruent to 2 modulo 3 and deduce a family of prime numbers in $\mathbb{Z}[\omega]$.
      iv. Find a number which is composite in $\mathbb{Z}[\omega]$ and is greater than 100 which is prime in $\mathbb{Z}$ and then find a composite integer which is congruent to 1 modulo 6 which doesn't appear as the norm of any Eisenstein integer (each unique in the class).
   (b) Working in your given "hextant" of the triangular grid, answer the following: [10]
      i. Calculate the norms of each of the numbers $s+t\omega$ with $\max(|s|, |t|) \leq 6$ and hence or otherwise classify them as prime or composite, giving the factors of any composite and explaining why any are prime.
      ii. Identify the elements of the ideal J generated by one of these primes and all of the cosets formed in $\mathbb{Z}[\omega]/J$.
      iii. Determine the ratio of the area of the smallest triangle made by elements of the ideal to the area of the fundamental triangle in $\mathbb{Z}[\omega]$ and relate it to the cardinality of the quotient ring.
      iv. Find all universal side divisors in your hextant of $\mathbb{Z}[\omega]$.