Math 4204 Assignment 2: Homomorphisms and Ideals

Answer all questions and show all working and check each of your results. Any rough work done before attempting your solutions should be attached to your answers as I need to know how you came up with them. You are allowed to talk with myself or other members of the class in general about the questions, but you must do them on your own.

The numbers represented by k and m should be replaced by the two largest digits of your registration number. Let me know which prime integer p congruent to 1 modulo 4 you choose, everybody must choose a different one.

1. Choose two non-trivial ideals I and J in $R := \mathbb{Z}_k \square \mathbb{Z}_m$ such that $I \subseteq J$. Identify the elements of the quotient rings $S := R/I$ and $T := J/I$ and then verify that $S/T \cong R/J$. Are either I or J prime ideals? [6]

2. (a) In $\mathbb{Z}_p[x]$, give an example of a principal ideal generated by a reducible polynomial and a non-principal ideal and in each case give a maximal ideal which contains them. Hence explain why only principal ideals generated by irreducible polynomials are maximal in $\mathbb{Z}_q[x]$ for any prime q. [3]

   (b) Suppose that M is a maximal ideal of a ring A and prove that A/M is a field by finding the inverse of a non-zero coset $M + a$ as follows; given that particular $a \in A$ consider the set $B := \{ax + y; x \in A, y \in M\}$ and prove that B is an ideal of A and hence deduce that the inverse coset exists. [6]

3. (a) Given the homomorphism ϕ from $\mathbb{R}[x]$ given by $\phi(f(x)) = f(i)$, find its kernel and image. Can we have homomorphisms between rings even if some of them are without unity? Give examples of each case that can exist. [3]

   (b) It is true that there exists $j \in \mathbb{Z}_q$ such that $j^2 \equiv -1 \pmod{q}$ for every prime $q \equiv 1 \pmod{4}$. Use this fact to factorise $j^2 + 1$ in $\mathbb{Z}[i]$ and hence show using norms that q is reducible in $\mathbb{Z}[i]$. Explain why this tells us that we can find integers c and d such that $q = c^2 + d^2$. Verify this fact and the steps of the proof for $q = p$. [6]