Math415 Graph Theory: Assignment 4 (December 2007)

Please show all working and reasoning to get full marks for any question. Hand in your rough working as well so I can see how you investigated and reached your final results. You are reminded that plagiarism is a serious offense and if it is detected you will be punished.

1. (a) Suppose you have a Hamiltonian graph G, prove that G must have an even number of vertices to allow it to have a perfect matching. Give an example of a non-Hamiltonian graph with 8 vertices and a Hamiltonian graph with 9 vertices. [3]

   (b) Explain why an even vertexed cubic Hamiltonian graph will contain three edge-disjoint perfect matchings. Find these matchings in the dodecahedron, the unique cubic graph in which every face is a pentagon. [2]

   (c) Consider some generalisations of this problem: [7]

      i. Show that for every $r \geq 2$ that K_{2r} and $K_{r,r}$ are regular Hamiltonian graphs which contain r edge-disjoint perfect matchings.

      ii. Prove that it is not true for any $r \geq 2$ that all $2r$-regular Hamiltonian graphs of vertices will contain $2r$ edge-disjoint perfect matchings.

      iii. Prove that an s-regular Hamiltonian graph doesn't even have to contain 2 edge disjoint Hamiltonian cycles if $s \geq 4$.

   (d) Verify that Grinberg's Theorem is satisfied in the dodecahedron. If the faces in a planar graph are only pentagons and at most two i-gons what possible values can i have? Give an example with $i > 5$. [4]

   (e) Prove that every 2-connected cubic graph has a perfect matching by showing such a graph must satisfy Tutte's condition. [5]
   [Hint: consider the sums of valencies in the odd components of $G - S$]

2. (a) What are $\delta(G \circ H)$ and $\beta(G \circ H)$ for a general G and H? [3]

   (b) What are $\alpha(G + H)$ and $\alpha'(G + H)$ in general? [3]

   (c) What criteria must be put on G and H for $G \times H$ to be Eulerian? If both G and H are connected does $G \times H$ have to be Hamiltonian? [3]

   (d) Prove that $\alpha'(G) \leq \frac{n}{2} \leq \beta'(G)$ for any n vertex graph G without any K_1 components. What bound can you prove for $\alpha(G)$ involving $\text{diam}(G)$? [3]