Math 4101 Assignment 4 April 2020

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. (a) Your selected graph H is planar, but not Hamiltonian. Use transitions to explain why it cannot be Hamiltonian, and then use Grinberg's equation to see whether or not it could potentially have a solution despite not being Hamiltonian. [2]

   (b) Find the independence and matching numbers of H and hence find a minimum vertex cover too. Find a cut-set of 3 vertices that gives some odd components on removal, but check that Tutte's 1-factor condition does hold and explain why it must for any set. [2]

2. (a) Finish the proof that any graph G with minimum valency δ must have a cycle of length at least $\delta + 1$ by considering a longest path in G and then consider which vertices a vertex at one end of the path can join to. Give an example of a graph (unique within the class) with $\delta \geq 3$ which has no cycle of length less than $\delta + 2$ [2]

   (b) As we did on page 60 of the notes for 7 vertices, continue to show that there is no hypo-Hamiltonian graph on either 8 or 9 vertices either. [4]

   (c) Suppose that M is not a maximal matching, but there is no alternating path with non-matched ends. Let N be a maximal matching and combine it with M to get a contradiction and hence prove the other direction of Theorem 4.6. [2]

3. (a) For your 8 vertex graph with 8 edges from assignment 3, calculate its set of vertex deleted subgraphs (its deck). Explain why the deck of the complement of any graph will contain the complement of each of the cards in the deck and hence how you would recognise a self-complementary graph from a deck. [2]

   (b) If you didn't know the number of edges or valency sequence of the graph, find two non-identical cards which could have come from a graph different from yours. [2]

   (c) Explain how, given the deck of any graph, you can always determine the graph's independence number. Give an example of a graph (unique within the class) which is not bipartite, but all graphs of its deck are. Explain how you can recognise when a deck comes from a bipartite graph. [3]

4. We will prove Hall's Theorem; for a bipartite graph G with partite sets A and B, there is a maximal matching of G if and only if $|S| \leq |N(S)|$ for all $S \subseteq A$.

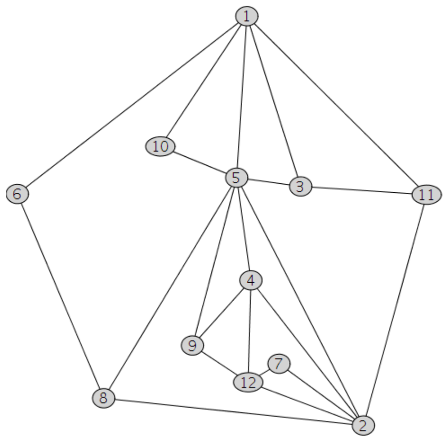
   (a) Firstly if $\mu(G) = |A|$ explain why $|S| \leq |N(S)|$ must be true for all $S \subseteq A$. [1]

   (b) Now we will prove the other direction by induction on $|A|$; establish it for $|A| = 1$. [1]

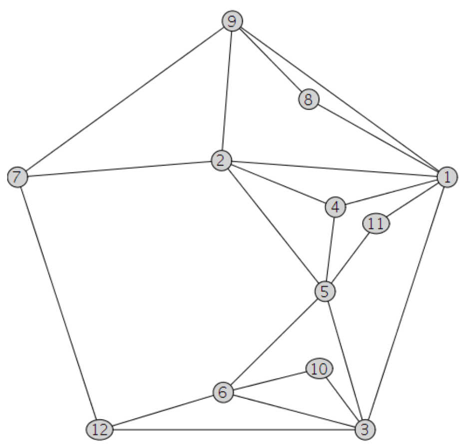
   (c) Pick any $a \in A$ and if you delete a and one of its neighbours $b \in B$, find the only condition under which induction on $G - \{a, b\}$ would not work. Let S be the subset of $A - \{a\}$ in question and determine the value of $|N(S)|$ in this case. [2]

   (d) Let $T := N(S)$ and explain why induction gives us a maximal matching between S and T, and then show why, for any $R \subseteq A - S$, there must be a matching that takes all vertices in R to $B - T$. [2]

Xucheng: $H_1 :=$



Mingxin: $H_2 :=$



Francis: $H_4 :=$