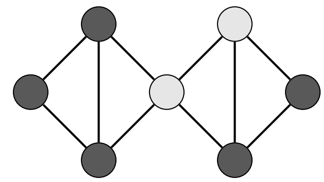
Math 4101 Assignment 3 March 2020

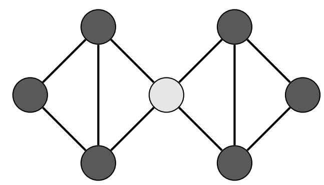
Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. You have selected a complement of a graph G at random.
   (a) Draw $\overline{G}$ and explain how, from any graph's complement, you can find the clique number of the original graph. Hence, or otherwise, find the chromatic number of G by giving a colouring with this number of colours and explaining why no colouring with fewer can exist. [3]
   (b) Verify that G has too many edges to be planar and apply the planar embedding algorithm until it fails. Using this, or otherwise, find a $K_{3,3}$ or K_5 (minor or homeomorph). [3]
   (c) Use $n - m + f = 0$ to predict how many faces G will have in the torus, and which face sizes can exist, and thus find an embedding of G on the torus. Show me all working, write what you are thinking and do not erase anything, I want to see more than the final drawing. [3]
   (d) With the embedding of $K_{3,3}$ or K_5 as a base, find a vertex v you can remove from G to enable $G - v$ to be embedded on the projective plane and do so, identifying faces of size larger than 3. [2]
2. (a) Given your chosen configuration of vertices of valency 5 and 6, complete the ring around them if they are to be a part of a planar triangulation. Find two colourings of the ring, one which can be immediately extended to the configuration, another which cannot, but just requires a Kempe chain interchange, and demonstrate that after that interchange we can colour the configuration. [2]
   (b) We want to prove that in any planar triangulation of minimum degree 5 there must exist a vertex of valency 5 adjacent to at least two vertices of valency 5 or 6. Suppose G is a graph in which this doesn't occur.
      i. Manipulate $n - m + f = 2$ to show that assigning charge of $14 - 2\rho(v)$ to each vertex v and -1 to each face gives a positive total charge. [1]
      ii. Using the three different configurations that can exist around a vertex of valency 5 in G show that discharging $\frac{4}{5}$ from each vertex of valency 5 to its neighbouring faces and $\frac{1}{3}$ from each vertex of valency 6 to its faces will give rise to a graph with negative charge. [2]
3. We want to prove that $\chi(G) \leq \Delta(G)$ for any connected graph G unless G is complete or an odd cycle. Let $n := |V(G)|$.
   (a) Explain why it suffices to consider $n \geq \Delta + 2$ and $\Delta \geq 3$. [1]
   (b) Suppose G has a cut vertex v. Use induction to prove that the result holds. [2]
   (c) Now suppose that G is 2-connected, but has a pair of vertices u and w which are not joined by an edge, but $G - \{u, w\}$ is disconnected. Let H_1 and H_2 be the components of $G - \{u, w\}$, and let $G_1 := G - H_2$ and $G_2 := G - H_1$.
      i. Why there must be at least one edge from u and w to both H_1 and H_2? [1]
      ii. Explain why it isn't always possible to colour G_1 and G_2 and combine them in a similar way as (b), but if, instead, we colour $J_1 := G_1 + uw$ and $J_2 := G_2 + uw$, it will be. Make sure to consider the cases when J_1 or J_2 are complete or odd cycles. [2]
   (d) If $G - \{u, w\}$ is connected for any choice of non-connected u and w, explain why, if y is a vertex of valency Δ, and u and w are neighbours of it, then we can make a list of the vertices of G as $[v_1 := u, v_2 := w, v_3, \ldots, v_{n-1}, v_n := y]$ such that v_{n-1} is a neighbour of v_n, v_{n-2} is a neighbour of either v_{n-1} or v_n, etc. Use the greedy colouring method on the list of vertices in G in the order from v_1 to v_n and deduce that G can be coloured using Δ colours. [3]

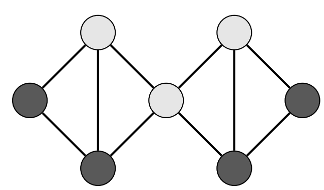
$$E(\overline{G_1}) := \{ac, ad, bf, bh, ce, cg, dh, gh\}$$



$$E(\overline{G_2}) := \{ab, ac, ad, bf, ce, cg, dh, eg\}$$



$$E(\overline{G_3}) := \{ac, ad, ah, bf, ce, cg, dh, ef\}$$



$$E(\overline{G_4}) := \{ac, ad, bf, ce, cg, dh, eh, fg\}$$