Math 4101 Assignment 2 February 2020

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. (a) Enter your graph into Maple, use IsIsomorphic to verify that it is self-complementary and find the eigenvalues as real numbers using evalf and Re. [1]

   (b) Pick one integer eigenvalue (not 0 or 1) of multiplicity at least 2 and use Pivot to find a basis for the eigenspace. Create an eigenvector not given by Maple and check that when you sum the numbers on the neighbours of each vertex that you do get the appropriate multiple of the original numbers as expected. Now use one of the non-integer eigenvalues and check the eigenvector for one vertex of each valency using decimals. [3]

   (c) If a graph G has adjacency matrix A, determine an expression for the adjacency matrix of $\overline{G}$ in terms of A, I and J and hence prove that, when G is a regular graph, the eigenvectors of G and $\overline{G}$ are the same, and their eigenvalues are related as in (d). [2]

   (d) Why does this mean that some pairs of eigenvalues of a self-complementary graph sum to -1? Which of your eigenvalues have this property? Explain how their eigenvectors are related to $\underline{j}$. [2]

2. (a) As we did for the 4-regular case, carefully show why either triangles or quadrilaterals must be formed if we try to create a 5-regular graph with 26 vertices and diameter 2. [3]

   (b) Explain why the maximum number of vertices that could be in a k-regular graph of diameter d is, for $k > 2$, $\dfrac{k\,(k-1)^d - 2}{k-2}$, and also find the answers for $k \leq 2$. [2]

   (c) Carefully adjust the method of counting from (b) to find how many vertices can be in such a graph if it is bipartite. [2]

3. Let us re-prove Turan's theorem in a different way. Suppose G is a maximal graph on n vertices that does not contain K_{r+1} and suppose the theorem is true for all graphs on fewer vertices.

   (a) Explain why $K := K_r$ must be a subgraph of G and hence find how many edges can be in G by counting the edges in K, $L := G - K$ (using the induction hypothesis) and between these two sets of vertices. [1]

   (b) Verify for $r = 3$ that the total number of edges in (a) is at most $T_{n,r}$ and explain why this will hold for any r, using the fact that $\frac{n-r}{r} = \frac{n}{r} - 1$. [2]

   (c) Generalise the complete bipartite analysis we did to find the eigenvectors (and hence the eigenvalues) of the general regular complete multi-partite graph $K_{m,m,\ldots,m}$. [3]

4. (a) Pick a triangle-free graph (unique within the class) with 6 vertices $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and 4 edges and create a 13-vertex graph from it as follows: add vertices u_j (for each v_j) and w, and add edges between w and each u_j and if $v_j v_k$ is an edge then add the edges $u_j v_k$ and $u_k v_j$. Verify that your 13-vertex graph has chromatic number 3 and no triangles. [2]

   (b) Explain why any graph with clique number 2 put into this process will give a graph with the same clique number but with chromatic number increased by 1. [2]

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M1:=             [1  1  1  1  0  0  0  0  0]
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                 [0  1  0  1  0  1  1  0  1]
                 [0  0  1  1  0  1  1  1  0]

                 [0  0  0  0  1  1  1  0  0]
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M3:=             [0  0  0  0  0  1  1  1  1]
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M4:=             [0  0  1  1  0  0  1  1  0]
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M5:=             [0  0  0  0  0  1  1  1  1]
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