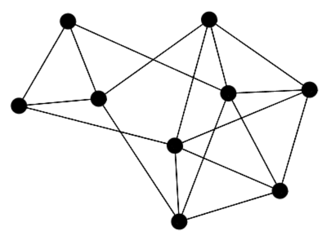
Math 4101 Assignment 1 January 2020

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. You have selected a list of edges of a graph L with 9 vertices $\{a, b, c, d, e, f, g, h, i\}$.
   (a) Find the valency sequence of L, carefully draw L trying to emphasize its symmetry and hence or otherwise find any vertices of eccentricity greater than 2 in L. [2]
   (b) Show that L is self-complementary, but find a reason why L is not isomorphic to this graph. [2]



   (c) Use the official Havel-Hakimi process to create a graph (unique within the class) with the same valency sequence as L, but explain why any graph created this way cannot be self-complementary. [2]
   (d) i. For any self-complementary graph G with $4j$ vertices, explain all the ways that a new vertex can be joined to $2j$ vertices in G such that the resulting graph is also self-complementary. [2]
   ii. Use your experience from the previous part to help you find which of the vertices of valency 4 can be deleted from L to leave a self-complementary graph. [1]
   (e) Given that H is a graph of diameter at least 4; prove that the diameter of H^c is at most 2. [2]

2. We are going to work through a proof of this result:
   " If G has at least $k + 1$ vertices and for all vertices u and v such that uv is not an edge we have $\rho(u) + \rho(v) \geq 2k$ then the average valency of G is at least k. "
   (a) Show how you went about creating 2 graphs unique within the class with 7 vertices and 13 edges, one which has $\rho(u) + \rho(v) \geq 6$ for all non-edges uv and the other which doesn't. [2]
   (b) Now suppose $|V(G)| = n$ and $|E(G)| = m$ and recalling Cauchy-Schwarz ($|\langle \underline{w}, \underline{z} \rangle| \leq ||\underline{w}|| \times ||\underline{z}||$), choose suitable vectors $\underline{w}$ and $\underline{z}$ to show that, in any graph G, $\sum_{v \in V(G)} ((\rho(v))^2) \geq \frac{4m^2}{n}$. [2]
   (c) If $\rho(u) + \rho(v) \geq 2k$ for all vertices u and v not joined by an edge, manipulate $\sum_{uv \in E(G)} (\rho(u) + \rho(v))$ to give an inequality linking $\sum_{v \in V(G)} ((\rho(v))^2)$, k, m and n. [2]
   (d) Combine the inequalities from (b) and (c) to show that $a(G) := \frac{2m}{n}$ must be at least k. [2]

3. Suppose that we have two graphs J_1 and J_2. Develop formulae, giving full explanations and examples of these properties of their products:
   (a) What is $\text{diam}(J_1 \Box J_2)$ in terms of $\text{diam}(J_1)$ and $\text{diam}(J_2)$? [2]
   (b) What is $\chi(J_1 \circ J_2)$ in terms of $\chi(J_1)$ and $\chi(J_2)$? [2]
   (c) How many triangles are there in $J_1 \times J_2$ in terms of the number of triangles in J_1 and J_2? [2]

$$E(L_1) := \{bc, bd, be, fg, fh, fa, ic, ig, ie, ia, cd, ca, gh, ge, dh, de, ha, ea\}$$

$$E(L_2) := \{bg, bh, be, fd, fh, fa, ic, ig, ih, ia, cd, ch, ce, ge, ga, de, da, ea\}$$

$$E(L_3) := \{bf, bh, be, fh, fa, ig, ih, ie, ia, cd, ch, ce, ca, gd, ge, ga, de, da\}$$

$$E(L_4) := \{bf, bd, be, fh, fa, ic, ig, ie, ia, cg, ch, ca, gd, ge, de, da, he, ha\}$$

$$E(L_5) := \{bf, bd, be, fh, fa, ic, ig, ie, ia, cg, cd, ca, gh, ge, de, da, he, ha\}$$