Math4101 Graph Theory: Assignment 4 (April 2012)

Please show all working and reasoning to get full marks for any question. Hand in your rough working as well so I can see how you investigated and reached your final results. You are reminded that plagiarism is a serious offense and when it is detected you will be punished.

1. (a) Prove by contradiction that any graph G has independence number $\alpha(G)$ and colouring number $\chi(G)$ satisfying: [2]

$$\alpha(G) \times \chi(G) \geq |V(G)|$$

   (b) Find two families of graphs unique within our class; one which has this relation as equality and another which is at least k greater than equality for any k. What are the matching numbers for your graphs? [4]

2. The domination number $\gamma(G)$ of a graph G is the cardinality of the smallest set of vertices such that every vertex is either in the set or is adjacent to a vertex in the set.

   (a) Characterise all graphs which have domination number 1. What can you say about the structure of bipartite graphs that have domination number 2? [3]

   (b) Explain why any maximal independent set (an independent set which can have no more vertices added to it without it becoming non-independent) must be a dominating set and also that the complement of that set is also a dominating set if the minimum valency is 1. What happens if there are vertices of valency 0? [4]

   (c) If a graph G has minimum valency at least 1 and maximum valency Δ then prove these inequalities and give graphs showing that both extremes can happen. [4]

$$2 \leq \frac{|V(G)|}{\gamma(G)} \leq 1 + \Delta$$

3. (a) Explain why any bipartite Hamiltonian graph must have both parts of equal sizes. Give a unique 2-connected bipartite *non-Hamiltonian* graph which has equal sized parts. If a Hamiltonian graph has chromatic number 3, what restrictions on the sizes of the colour classes exist? [4]

   (b) Under what conditions is $(G + H)$ an Eulerian or a Hamiltonian graph? Can it be both or neither? [4]