Math4101 Graph Theory: Assignment 3 (March 2012)

Please show all working and reasoning to get full marks for any question. Hand in your rough working as well so I can see how you investigated and reached your final results. You are reminded that plagiarism is a serious offense and when it is detected you will be punished.

1. Prove by induction on the number of edges in a graph that any bipartite graph has edge colouring number equal to its maximum valency. Find such an edge colouring for a bipartite 4-regular cartesian or tensor product of your choice of 2-regular graphs. [5]

2. (a) Create a graph with 8 vertices and 21 edges by choosing a tree on 8 vertices (with diameter between 4 and 5 and unique within the class) and taking its complement for your graph G. [1]

   (b) Explain why the diameter of almost every complement of a tree will be 2 and find all exceptions to this rule. [2]

   (c) Calculate what combinations of face sizes G could have on the torus. [1]

   (d) Explain algebraically why G cannot be planar and find a Kuratowski subgraph in it. What is the chromatic number of G? [3]

   (e) carefully embed G on the torus, explaining your thoughts at each step. Verify your face sizes match your prediction by listing all faces. [3]

3. (a) Manipulate Euler's formula to show that in any planar triangulation with minimum valency 5 and n_j vertices of valency j that [2]

$$4 = 2f - \sum_{j\geq 5}(j-2)n_j$$

   (b) Find a second relation linking f and $\sum_{j\geq 5} n_j$ to prove that we have [3]

$$-f + 4n_5 + 2n_6 - 2n_8 - 4n_9 - \ldots = 28$$

   (c) Give charge of -1 to each face, 4 to each vertex of valency 5, 2 to each vertex of valency 6, etc. as prescribed above. Our discharging rule will be to give all charge from every vertex of valency 5 and 6 equally to their neighbouring *faces*.

   Under the assumption that any vertex of valency 5 has at most one neighbour of valency 5 or 6 prove that in each possible case the charge on the graph must become negative. Explain what this implies for the four colour theorem. [5]