Math4101 Graph Theory: Assignment 2 (February 2012)

Please show all working and reasoning to get full marks for any question. Hand in your rough working as well so I can see how you investigated and reached your final results. You are reminded that plagiarism is a serious offense and when it is detected you will be punished.

1. Prove that if two vertices v and w are joined by an edge then $|\varepsilon(v) - \varepsilon(w)| \leq 1$, by contradiction. [3]

2. Suppose G_1 and G_2 are bipartite graphs. You can choose examples for them to test the properties and show that you understand but should prove the properties for any graphs.

   (a) Explain why $G_1 \square G_2$ (the cartesian product) is bipartite and this is the only way you can get a cross product to be bipartite. [2]

   (b) Prove that $G_1 \times G_2$ (the tensor product) is both bipartite and disconnected. Explain how a closed walk of odd length in G_1 could lead to connectedness of the tensor product. [4]

   (c) Under what circumstances could $G_1 \circ G_2$ (the corona product) be bipartite? [2]

3. Use the block form of the adjacency matrices of bipartite graphs to explain why if λ is an eigenvalue of a bipartite graph then $-\lambda$ is also an eigenvalue with a closely related eigenvector. Choose a small bipartite graph not used in class or chosen by a classmate and find its eigenvalues and eigenvectors and check that this relation holds. [5]

4. (a) Create a 4-regular graph (unique within the class) with 9 vertices and determine whether or not it is self-complementary. [2]

   (b) By considering complements count the number of non-isomorphic 4-regular graphs with less than 9 vertices. [2]

   (c) Explain why not every 4-regular graph with n vertices can be formed from one with $n-1$ vertices by removing two edges with no vertices in common and adding four edges replacing the two which were removed to a new vertex; find a unique example with more than 6 vertices for which no vertex can be removed without creating a multiple edge in the smaller 4-regular graph. Find a way to reduce your graph by removing 2 vertices and rejoining edges and explain for which graphs it would fail. [5]