Math 3205 Assignment 1 September 2019

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. Do not erase anything. Include all rough work and do not commit plagiarism. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

1. The Catalan numbers are defined as follows:

$$C_j := \frac{1}{j+1}\binom{2j}{j}$$

   (a) Carefully expand the factorial form of these binomial coefficients to show that: [4]

$$C_m = \binom{2m+1}{m+1} + \binom{2m}{m+2} - \binom{2m+1}{m+2} - \binom{2m}{m+1}$$

   (b) Identify the relative positions of the binomial coefficents in (a) in the Pascal matrix and verify the relation for a value of $m \geq 3$ which is different than that chosen by anyone else in the class. [2]

   (c) Explain how (a) can be used to prove that $C_j \in \mathbb{Z}$ for all positive integers j. [1]

2. Given a set of cardinality n, and a permutation ρ, we say that ρ has r fixed points if r of the n elements are mapped to themselves; i.e. $\rho(l) = l$ for r different elements l. Let $F(n, r)$ count the number of permutations on n elements with r fixed points.

   (a) Showing all your working and explanations, create the matrix $F(n, r)$ for n and r from 0 to 5. [2]

   (b) i. Explain why $F(n+1, 0) = n \times (F(n, 0) + F(n-1, 0))$ using an analogy unique within the class. [2]

      ii. Use strong induction to show that the relation in i. gives, for any positive integer t: [3]

$$F(t, 0) = t! \times \sum_{k=0}^{t} \frac{(-1)^k}{k!}.$$

      iii. Use ii. to determine what real number $\lim_{s\to\infty} \frac{F(s, 0)}{s!}$ is equal to. [1]

   (c) Experiment and try to develop a combinatorial argument to find a recursive formula for $F(n+1, r)$ for any $r \geq 1$ in terms of $F(n, s)$ with $r - 1 \leq s \leq r + 1$. [4]