Chromatic Polynomials

We define the function $P(G,t)$ as the number of ways in which one can colour the vertices of G using t colours, where each vertex is labeled so that any problems with isomorphism are suppressed. For instance, $P(K_3,2)=0$ but $P(K_3,3)=6$. If $|V(G)|=n$ then $P(G,t)$ is a monic polynomial of degree n.

By inspection we can note that $P(K_n,t)=\frac{t!}{(t-n)!}$(since we have t colours available for the first vertex, the next can choose any one of the $t-1$ others, and this continues since every vertex is adjacent to every other. Similarly, $P(\overline{K_n},t)=t^n$ since every vertex can have any of the t colours. For any tree T we have $P(T,t)=t(t-1)^{n-1}$ since the first vertex can have any of the t colours and we choose the colours of the remaining vertices recursively by considering neighbours of those already coloured. We always have $t-1$ colours to choose from since any vertex has exactly one previously coloured neighbour as there are no circuits.

In order to calculate chromatic polynomials there are several very useful more advanced techniques:

1. If $G=G_1 \cup G_2$ (in that every edge of G is in either G_1 or G_2 or perhaps both) and $G_1 \cap G_2 = K_i$ then we use Complete Intersection:

$$P(G,t)=\frac{P(G_1,t)P(G_2,t)}{P(K_i)}$$

2. If e is an edge of G then we have Deletion-Contraction:

$$P(G,t)=P(G-e,t)-P(G\circ e,t)$$

   or equivalently, if there is no edge e, Addition-Identification:

$$P(G,t)=P(G+e,t)+P(G\circ e,t)$$

   ($G\circ e$ is the graph formed by taking the two vertices of e and coalescing them together into one vertex, only removing multiple edges)

3. Finally, if v is adjacent to every vertex of G then

$$P(G,t)=t\times P(G-v,t-1)$$