Math2101 Handout 7: Graphs

We have a set V of *vertices*, which are joined by a set E of *edges* between pairs of different vertices forming a graph G. We usually do not put labels on our vertices so do not distinguish between two graphs that are essentially the same apart from the labels.

- **Valency** The *valency* of a vertex is the number of edges at it. The *valency sequence* is the valencies of each vertex arranged in non-increasing order and it tells us quite a lot about the graph. However, two graphs can be different and have the same valency sequence.

- **Connectivity** We can move around the graph from vertex to vertex using the edges, as if they were towns and roads. Define a graph as *disconnected* if we cannot move along the edges to get from some vertex to another vertex, and *1-connected* if it is not disconnected, but there is a vertex which can be removed with its edges so that the remaining graph is disconnected.

- **Isomorphism** We say that two graphs are *isomorphic* if there exists a 1-1 and onto function between the two sets of vertices which preserves all the edges. Normally we can see it more clearly by redrawing one graph to look like the other.

  For these two graphs they both have valency sequence $(4,3,3,2,2,1,1)$. However, G and H are not isomorphic. We can see that the vertex of valency 4 is a cut-vertex in both graphs but its removal splits the remainder of the graph into different sized subgraphs. Also note that it is not adjacent to a vertex of valency 1 in H.

- **Cycles** A *cycle* in a graph is a sequence of edges which returns to the start point without repeating any other vertices. If we can move through every vertex exactly once and return to the start the graph is *Hamiltonian*. It is difficult to tell when a graph is not Hamiltonian, but to show it is we can just demonstrate a cycle. If we can move along every edge exactly once (repeating vertices if necessary) and return to the start, the graph is *Eulerian* and this is true if and only if each vertex is of even valency.

- **Colouring** We can colour the vertices of a graph such that no edges joins two vertices of the same colour, and the smallest number of colours necessary for this is called its *colouring number*. Any graph without odd cycles can be proven to have colouring number at most two and, famously proven in the 1970s, any graph which can be drawn without any edges crossing must have colouring number at most four.

- **Special Graphs:** The Cycle graph C_n is the unique graph on n vertices which is connected and has all vertices of valency 2. P_n is formed from C_n by removing any edge, forming the Path graph on n vertices. K_n is the Complete graph on n vertices in which every vertex is joined to every other vertex; it has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges. Given any graph G we can create the complement of G as the graph which has the same vertex set as G but just the edges which G does not have.