Math2101 Handout 5: Relations

- **Relations:** We say that a set of ordered pairs (s, t), where $s \in S$ and $t \in T$ is a relation which is:

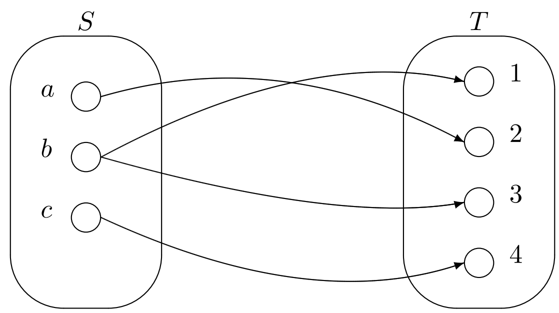
  Everywhere Defined: if there is a pair of the form (s, y) for all $s \in S$ for some $y \in T$.

  Onto: if there is a pair of the form (x, t) for all $t \in T$ for some $x \in T$.

  Uniquely Defined: if there are no two pairs both of the form (s, y) for any $s \in S$.

  One to One: if there are no two pairs both of the form (x, t) for any $t \in T$.

  A relation can be shown in several forms, graphical:



  or in set form: $R := \{(a, 2), (b, 1), (c, 4), (b, 3)\}$, and in tabular form:

R	1	2	3	4
a		✓		
b	✓		✓	
c				✓

  This R is a relation from $S := \{a, b, c\}$ to $T := \{1, 2, 3, 4\}$ which is e.d. but not u.d. since there are two arrows from b. R is onto and 1-1 since there is exactly one arrow into each element in T.

  – A *function* is a relation which is both uniquely and everywhere defined.

  – The *inverse of a relation* is the set of reversed ordered pairs (t, s). Note that the inverse of a function is a function only if it is onto and 1-1.

  A relation R between a set S and itself is *reflexive* if $(s, s) \in R$ for all $s \in S$, it is *symmetric* if $(s, t) \in R$ implies $(t, s) \in R$ and *transitive* if $(s, t) \in R$ and $(t, u) \in R$ implies $(s, u) \in R$. A relation is *anti-symmetric* if $(s, t) \in R$ implies $(t, s) \notin R$ (for $s \neq t$). A relation is *anti-reflexive* if $\forall s \in S; (s, s) \notin R$. Many common mathematical relationships can be modelled using such a relation.

  – A relation which is reflexive, symmetric and transitive is an *equivalence relation*. An example of this is "$s - t$ is even" on the integers which breaks them into two groups; odds and evens.

  – A relation which is reflexive, anti-symmetric and transitive is a *partial order*. An example of a partial order is "$A \subseteq B$" on sets A and B.

  – A relation which is anti-reflexive and symmetric is a *graph*; we will study these objects later.

  – A permutation is a 1-1 and onto function between a set and itself, such as $\{(a, c), (b, b), (c, d), (d, a)\}$.

  Two relations can be composed with each other if the output set of one is the same as the input set of the other; that is, if R maps from S to T and Q maps from T to U then there exists a composition of relations $Q \circ R$ which maps from S to U formed by following all sequences of two arrows in the combined diagram. Note the functional notation is used since we consider the possible relation of s to $Q(R(s))$ for all elements $s \in S$. This will always work for a relation from one set to itself, of course.

Math2101 Handout 6: Combinatorics

- **Pigeonhole Principle:**

  Given a set of p objects and h groups to place them in, if $h < p$ then at least one group has two objects in it. In general, we can say that one group must have *at least*

  $$\left\lceil \frac{p}{h} \right\rceil$$

  objects in it (those straight brackets round up the nearest integer).

  Note that it can sometimes be useful to round down, in which case the formula is $\left\lfloor \frac{p-1}{h} \right\rfloor + 1$

- **Counting:**

  There are two fundamental rules of counting; multiplicative and additive.

  - *Multiplicative*: if we have two events that are independent of each other and we want to know how many ways both can happen together, we multiply the number of ways each can happen. For instance, to roll a 6-sided die and flip a coin there are twelve outcomes of the form $(d.c)$ where $d \in \{1, 2, 3, 4, 5, 6\}$ and $c \in \{\text{heads}, \text{tails}\}$ and there are $12 = 6 \times 2$ possible such pairs.
  - *Additive*: If we have two events where only one of the two can happen then the total number of events is the sum of the number of ways each event can happen. For instance, if we are adopting one animal from the SPCA and there are 5 dogs and 8 cats at the shelter we have $13 = 5 + 8$ possible choices of a new pet.

  The most general cases of counting involve choosing r objects from a pool of n objects under certain conditions. We have formulae for each of these combinations of conditions;

  - *Repetition*: whether or not we can draw "the same object" again once it has been drawn or not, as if we are replacing the object back in the pool of n objects each time we pick. Note: if we have multiple objects which are indistinguishable and we aren't replacing them, we are still in this case, *assuming* we aren't going to run out of these similar objects.
  - *Order*: whether or not we are picking all r objects at once or in a sequence, such as drawing a hand of cards or picking a soccer team by position.

	Repetition Allowed	Repitition Forbidden
Order Important	n^r a sequence of r rolls of a die n is number of faces of the die	${}_nP_r = \dfrac{n!}{(n-r)!}$ r lotto balls in order n is number of balls
Order Unimportant	$\binom{r+n-1}{r} = \dfrac{(r+n-1)!}{r!(n-1)!}$ r arrows, how many in each segment? n is number of segments	$\binom{n}{r} = {}_nC_r = \dfrac{n!}{r!(n-r)!}$ hand of r cards n is number of cards