Math2101 Handout 4: Proof by Induction (2019)

- **Induction:** Given a statement $p(n)$ about an integer n we wish to show it is true for all integer values of n at least a and we proceed as follows:

  - *Initial Case:* Show that $p(a)$ is true
    (optionally also test $p(a+1)$ and $p(a+2)$ to see how the induction will proceed).
  - *Inductive Case:* Assume $p(k)$ is true for some value of $k \geq a$. State one side of $p(k+1)$ in terms of the corresponding side of $p(k)$ and use the assumptions to deduce that the other side of $p(k+1)$ is related to the original side in the same way as $p(n)$ was.

  For example, the sum of the first n square numbers has a nice formula:

  $$p(n) :\equiv \text{“}\sum_{j=1}^{n} j^2 := 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}\text{”}$$

  - *Initial Case:* The first possible value of n is 1, so we consider $p(1) :=$ “$1^2 = 1 = \frac{1\times(1+1)\times(2\times1+1)}{6} = 1$” as required. Similarly, $p(2) :=$ “$1^2+2^2 = 5 = \frac{2\times(2+1)\times(2\times2+1)}{6} = 5$” and $p(3) :=$ “$1^2+2^2+3^2 = 5+3^2 = 14 = \frac{3\times(3+1)\times(2\times3+1)}{6} = 14$”.
  - *Inductive Case:* Assume $p(k) :=$ “$\sum_{j=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$”. Now the left hand side (LHS) of $p(k+1)$ is

    $$\sum_{j=1}^{k+1} j^2 = \left(\sum_{j=1}^{k} j^2\right) + (k+1)^2 = \text{LHS}(p(k)) + (k+1)^2.$$

    But using the assumption (the inductive hypothesis), we get that

    $$\begin{aligned}
    \sum_{j=1}^{k+1} j^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
    &= (k+1)\frac{k(2k+1)+6(k+1)}{6} \\
    &= (k+1)\frac{(2k^2+7k+6)}{6} \\
    &= (k+1)\frac{(2k+3)(k+2)}{6} \\
    &= \frac{(k+1)(k+2)(2k+3)}{6} \\
    &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
    \end{aligned}$$

    But this is exactly the statement $p(k+1)$ that we wished to establish!

- **Basic formulae in Sigma Notation:**

$$1+\ldots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad b+\ldots+b = \sum_{i=1}^{n} b = nb \qquad 1+x+\ldots+x^n = \sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$