Math2101 Handout 3: Methods of Proof

We shall show how to go about proving the statement "if an odd integer is multiplied by -1 and that new integer is then added to 26 the result is odd". We first identify the statements $p(x)$ and $q(x)$ in the above statement if it is $p(x) \longrightarrow q(x)$ and deduce that:

$p(x) \quad :\equiv \quad$ "x is odd " $\quad \equiv \quad$ "$x = 2j + 1$ for some $j \in \mathbb{Z}''$

$q(x) \quad :\equiv \quad$ "$26 - x$ is odd" $\quad \equiv \quad$ "$26 - x = 2k + 1$ for some $k \in \mathbb{Z}$"

We are usually either told to use one of the methods below, or we can choose one:

- **Direct:** We suppose that $p(x)$ is true and using what that tells us about x we then apply that to the subject of $q(x)$ in order to try to show that it is true when $p(x)$ is.

  So if $p(x)$ is true then $x = 2j + 1$, and $q(x)$ is about $26 - x$, and putting these two things together

  $$26-x = 26-(2j+1) = 26-2j-1 = 25-2j = 25+2\times(-j) = 1+24+2\times(-j) = 2\times(12-j)+1$$

  Thus we have our statement in the form of $q(x)$ where $k = 12 - j$, and it just remains to establish that this k is an integer. Since j is, multiplying by -1 means $-j$ is still an integer, and then subtracting this from 12, another integer, means that k is an integer as required.

- **Contrapositive:** We can alternatively suppose that $q(x)$ is false and use what that tells us about x to apply it to the subject of $p(x)$. We are thus aiming to prove that $(\sim q(x)) \longrightarrow (\sim p(x))$ which we have shown is logically equivalent to $p(x) \longrightarrow q(x)$.

  So if $q(x)$ is false we use the fact that, for integers, not being odd is the same as being even, so $\sim q(x)$:≡"$26 - x = 2m; m \in \mathbb{Z}$", and $\sim p(x)$ says that "$x = 2n; n \in \mathbb{Z}$". Simplifying $(\sim q(x))$:

  $$\begin{aligned} 26 - x &= 2m \\ x &= 26 - 2m \\ &= 2 \times (13 - m) \end{aligned}$$

  This equals $2n$ if we take $n = 13 - m$ and so, again, since m is an integer, so is $-m$ and adding 13 to this keeps it an integer, so n is an integer as required.

- **Contradiction:** We now suppose that $p(x)$ is true and also that $q(x)$ is false. We intend to get an impossible situation arising whence we can use the logical equivalence of $(p(x) \wedge (\sim q(x))) \leftrightarrow (\sim T_0)$ and $(p(x) \longrightarrow q(x)) \leftrightarrow T_0$ to show that $p(x)$ implies $q(x)$ as required.

  As before, if $p(x)$ is true then $x = 2j + 1$, and $(\sim q(x))$ says that $26 - x = 2m$, for integers j and m. Combining these two statements to remove x we get:

  $$\begin{aligned} 26 - (2j + 1) &= 2m \\ 25 &= 2j + 2m = 2(j + m) \\ \frac{25}{2} &= j + m \end{aligned}$$

  This statement is our desired contradiction since both j and m are integers and so their sum is an integer, but $\frac{25}{2}$ is certainly not an integer as it is between 12 and 13 and there is no integer in that gap since $13 = 12 + 1$.