Math2101 Handout 3: Methods of Proof

We shall show how to go about proving the statement "if an odd integer is multiplied by -1 and that new integer is then added to 26 the result is odd". We first identify the statements $p(x)$ and $q(x)$ in the above statement if it is $p(x) \to q(x)$ and deduce that:

$$\begin{array}{lclcl} p(x) & :\equiv & \text{"}x\text{ is odd "} & \equiv & \text{"}x = 2j+1 \text{ for some } j \in \mathbb{Z}'' \\ q(x) & :\equiv & \text{"}26 - x \text{ is odd"} & \equiv & \text{"}26 - x = 2k+1 \text{ for some } k \in \mathbb{Z}\text{"} \end{array}$$

We are usually either told to use one of the methods below, or we can choose one:

- **Direct:** We suppose that $p(x)$ is true and using what that tells us about x we then apply that to the subject of $q(x)$ in order to try to show that it is true when $p(x)$ is.

  So if $p(x)$ is true then $x = 2j + 1$, and $q(x)$ is about $26 - x$, and putting these two things together

  $$26-x = 26-(2j+1) = 26-2j-1 = 25-2j = 25+2\times(-j) = 1+24+2\times(-j) = 2\times(12-j)+1$$

  Thus we have our statement in the form of $q(x)$ where $k = 12 - j$, and it just remains to establish that this k is an integer. Since j is, multiplying by -1 means $-j$ is still an integer, and then subtracting this from 12, another integer, means that k is an integer as required.

- **Contrapositive:** We can alternatively suppose that $q(x)$ is false and using what that tells us about x we then apply that to the subject of $p(x)$ in order to try to show that it is false when $p(x)$ is. We are thus proving $(\sim q(x)) \to (\sim p(x))$ which we know is logically equivalent to $p(x) \to q(x)$.

  So if $q(x)$ is false we use the fact that, for integers, not being odd is the same as being even, so "$26 - x = 2m; m \in \mathbb{Z}$", and $\sim p(x)$ says that "$x = 2n; n \in \mathbb{Z}$". Simplifying $(\sim q(x))$:

  $$\begin{aligned} 26 - x &= 2m \\ x &= 26 - 2m \\ &= 2 \times (13 - m) \end{aligned}$$

  This equals $2n$ if we take $n = 13 - m$ and so, again, since m is an integer, so is $-m$ and adding 13 to this keeps it an integer, so n is an integer as required.

- **Contradiction:** We now suppose that $p(x)$ is true and also that $q(x)$ is false. We intend to get an impossible situation arising whence we can use the logical equivalence of $(p(x) \wedge (\sim q(x))) \leftrightarrow (\sim T_0)$ and $(p(x) \to q(x)) \leftrightarrow T_0$ to show that $p(x)$ implies $q(x)$ as required.

  As before, if $p(x)$ is true then $x = 2j + 1$, and $(\sim q(x))$ says that $26 - x = 2m$. Combining these two statements to remove x we get:

  $$\begin{aligned} 26 - (2j + 1) &= 2m \\ 25 &= 2j + 2m = 2(j + m) \\ \frac{25}{2} &= j + m \end{aligned}$$

  This statement is our desired contradiction since both j and m are integers and so their sum is an integer, but $\frac{25}{2}$ is certainly not an integer as it is between 12 and 13 and there is no integer in that gap. Also it is 12.5 in decimal terms and no integer has to be written with a decimal point.