Math 2101 (2018/19) Workshop 5: Graph Theory

1. (a) The graphs with 6 vertices and 4 edges are the same as those with 5 vertices (with an extra vertex of valency 0 added) apart from 3; find them!
   (b) Get as many of the 15 graphs with 6 vertices and 5 edges as you can, by building them from part (a), using symmetry.
   (c) Identify the valency sequences, cut vertices and colouring numbers of these graphs, to help you decide if you have any duplicates. If you do, show they are isomorphic by labelling them or redrawing them to show they are the same.
   (d) *There are two different graphs with 6 vertices and 9 edges such that all vertices are of valency 3. Deduce what their complements must be and hence find them.
   (e) *Which of the graphs or their complements from (d) are Eulerian and/or Hamiltonian?

2. The distance between two vertices u and v is the smallest number of edges in a sequence $uv_1v_2 \dots v$, the eccentricity of a vertex is the largest distance to any other vertex in the graph. A graph has diameter d if all of its vertices have eccentricity d or less. A cubic graph has all vertices of valency 3.
   (a) Pick two connected graphs from Q1(b) and find the eccentricities of their vertices. Note that there is only one direct path between any pair of vertices since there will be no cycles in your graphs.
   (b) Explain why a graph can only have diameter 1 if it is a complete graph.
   (c) Find two cubic graphs with 8 vertices which are non-isomorphic but which have every vertex of eccentricity 3.
   (d) *Show that the diameters of the two non-isomorphic cubic graphs with 6 vertices from Q1(d) are different.
   (e) *Determine the unique cubic graph with 10 vertices and diameter 2 by fixing one vertex and its 3 neighbours and then joining the other 6 vertices in the only way that does not create any triangles or quadrilaterals.
   (f) *Explain why there cannot exist a larger cubic graph with diameter 2 by showing that if all vertices are valency k then the maximum number of vertices in such a graph is $k^2 + 1$. Try to logically create a graph with 17 vertices of valency 4 with diameter 2, but indicate when things fail for you.

3. The graph which dictates who works with who in the workshops has 9 vertices, one for each student.
   (a) Each week the graph is made from one triangle K_3 and 3 pairs of vertices as K_2. Show that you can take 3 copies of this such that everyone is in the triangle once by choosing the week 1 picture for a circle of 9 vertices and then rotating it around 1/3 of a circle twice.
   (b) Explain why we can pick the K_2s such that they join vertices at different distances greater than 1.
   (c) If we take the graphs of the first 3 weeks together, determine what its valency sequence must be, and hence that it could be self-complementary.
   (d) *Carefully draw the complement of the graph from (c) and find an isomorphism which shows that it actually is self-complementary, and hence everyone was in a group with everyone else exactly once!