Math 2101 (2018/19)
Workshop 2: Logic and Proofs

We are going to investigate logic and simple proofs this time:

1. (a) Using squared paper, or Maple, carefully plot the curves $x^2 + y^2 = 4$, $y = \frac{x}{2} + 1$ and $y^2 = 3x - 4$ for x in the range $[-3, \ldots 6]$.

   (b) Now make a new diagram using (a) and shade in the different regions of $\mathbb{R}^2$ where the following logic statements are true:
   $p(x, y) :\equiv$ " $x^2 + y^2 < 4$ " , $q(x, y) :\equiv$ " $3x - 4 \geq y^2$", $r(x, y) :\equiv$ " $\left|y - \frac{x}{2}\right| \leq 1$ ".

   (c) i. Identify the regions of $\mathbb{R}^2$ where $(p(x, y) \wedge (\sim r(x, y)))$ is true.

   ii. Find how many points could be used to show $\exists x, y \in \mathbb{Z} : (p(x, y) \wedge r(x, y))$, and verify the points using algebra.

   iii. Similarly explain why $\exists x, y \in \mathbb{Z} : (p(x, y) \wedge q(x, y))$ is false.

   iv. Are there an infinite number of integer points where $(q(x, y) \wedge r(x, y))$ is true?

   (d) Similarly, determine whether or not the following statements are true or false:

   i. $\forall x, y \in \mathbb{Z} : (p(x, y) \wedge q(x, y)) \longrightarrow r(x, y)$

   ii. $\exists x, y \in \mathbb{R} : (p(x, y) \wedge q(x, y)) \longrightarrow r(x, y)$

   iii. $\forall x \in \mathbb{R} : (\ \exists y > 0 : \sim r(x, y))$

   iv. $\exists y \in \mathbb{Z} : (\ \forall x > 0 : p(x, y) \vee q(x, y))$

2. Let $w(n) :\equiv$ "n is odd$''$ $\equiv$ " $n = 1 + 2 \times j : j \in \mathbb{Z}$".

   (a) Explain why "$\forall m \in \mathbb{Z} : (w(m) \vee (\sim w(m)))$" is a true statement and explain exactly when $(\sim w(m))$ is true with this domain.

   (b) Prove that, for integers n, if n is odd then n^2 must be odd, and so $w(n) \longrightarrow w(n^2)$.

   (c) Create the truth table for $(q \vee (\sim p))$ and compare it to that for $p \longrightarrow q$ and $(\sim q) \longrightarrow (\sim p)$.

   (d) Express $(\sim w(n^2))$ and $(\sim w(n))$ in words and hence deduce something about integers, even numbers and squares.

   (e) Why isn't (d) true if $n \in \mathbb{R}$?

   (f) If $a^2 = 2 \times b^2$ for integers a and b use (d) to explain why both a and b must be even, and see what happens after you assume that.