Math 2101 (2016/17)
Workshop 3: Different Induction Techniques

1. The *power set* of a set is the set of all subsets of it.

   (a) Pick any set A of containing two different positive numbers and explain why there are 4 different subsets of A.

   (b) Add one new element to A and explain why there are now 8 subsets, using the whether or not the new element is in to divide the subsets into two cases.

   (c) Similarly prove by induction that there are $2^{|S|}$ subsets for any set S.

2. An positive integer p is *prime* if $q > 1$ such that $p = qk$ then $q = p$ (where $q, k \in \mathbb{Z}$).

   (a) Explain which of the numbers 11 to 15 are prime.

   (b) Use strong induction to prove that every positive integer is equal to a product of prime numbers.

   (c) Why would ordinary induction not work for this induction proof?

3. The arithmetic mean and the geometric mean of a set of non-negative numbers satisfy:

$$P(n) :\equiv \text{“}\frac{a_1 + a_2 + \ldots + a_n}{n} = \frac{\sum_{j=1}^{n} a_j}{n} \geq \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} = \sqrt[n]{a_1 \times a_2 \times \ldots a_n}\text{”}$$

   (a) Verify that this is true for your sets of cardinality 2 and 3 from question 1(a) and then prove by contradiction that it is true for any set of cardinality 2, trivially true for any set of cardinality 1 and equal when all elements in the set are equal.

   (b) We are going to use forward-backward induction for this relation, proceeding as follows. Show that $P(k) \longrightarrow P(2k)$ and $P(m) \longrightarrow P(m-1)$ will indeed show that $P(n)$ is true for all n at least your initial value.

   (c) By taking a generic set of $2k$ numbers and breaking it into 2 equally sized sets show that $P(k) \longrightarrow P(2k)$.

   (d) Now suppose $P(m)$ is true and then consider the special case when $a_m := \frac{1}{m-1}\sum_{j=1}^{m-1} a_j$.

   By careful substitution and powering show that this implies that $P(m-1)$ is true and hence this completes the backward part of the induction.