Cape Breton University

MATH2101

Discrete Mathematics

14th December 2011 — Time : 3 hours

Clearly write your answers to the questions showing all reasons, working and checks and indicate what each mathematical calculation is doing. The marks from 5 questions will give your final mark. Do not erase anything and start each answer on a fresh piece of paper. Feel free to write explanations of what you are thinking at each stage, nothing you can write can lose you marks!

Q1. We will be investigating this proposition:

$$p(n) :\equiv \text{“}\sum_{j=0}^{n}((j+1)\times 3^j) = \frac{(2n+1)\times 3^{n+1}+1}{4}\text{”}$$

(a) Verify that $p(n)$ is true for $n = 0$, 1 and 2, and explain why also for $n = -1$. [3]

(b) Prove $p(n)$ is true for all $n \geq 0$ using induction. [6]

(c) Manipulate $p(n)$ to find a simplified expression for $\sum_{j=m}^{n}((j+1)\times 3^j)$. [3]

Q2. (a) Create a graph G with valency sequence (4,3,3,2,2,2) with connectivity 1. Find four different Eulerian trails in G, explaining why they are different (not backwards versions) and also why there cannot be a Hamiltonian cycle in G. [4]

(b) Create a different graph H with the same valency sequence as G and draw it without any edges crossing. Check that the deletion of any one of H's vertices leaves a connected graph. Use these subgraphs to identify two vertices which are symmetrical in H, if possible. [5]

(c) Can you find a pair of vertices in H whose deletion leaves a graph with valency sequence (1,1,1,1)? If not explain why it is impossible and if so, find a colouring with the fewest colours given that these two vertices are coloured with different colours, explaining why. [3]

Q3. (a) Simplify the logic expression $(p \to q) \wedge (q \vee p)$. [5]

(b) Create a logic expression containing 3 different letters in two different bracketed expressions with all different logical operators that simplifies to a single letter. [3]

(c) Explain why $\to$ is not an associative operator using algebra and truth tables. [4]

Q4. On a breakfast table are the following food items; 5 eggs, 7 sausages, 4 pieces of toast, 2 oranges and a grapefruit. For each answer explain what method of counting you are using and why.

(a) If there are 5 people at breakfast and all the items are eaten, what is the smallest number of items that one particular person might have eaten? How many people must have eaten more than 3 items? How many people could have eaten more than 4 items? [3]

(b) How many plates containing different food items were initially possible? How many ways could someone have had two (not necessarily distinct) items of food on their plate? [4]

(c) If the 5 people had seen the initial arrangement and chosen one food item as their favourite, how many difference sets of choices could there have been so that not everyone got their favourite food? [5]

Q5. Let $q(x) :\equiv$ " $x^3 + 7x^2 > 36$ " and $r(x) :\equiv$ " $|4x + 6| \leq 7$ ".

(a) Plot the values for which $q(x)$ and $r(x)$ are true on two real lines by solving appropriate equations. [4]

(b) Are the following statements true or false? Give full reasons for your answers: [8]

$$\begin{aligned} \forall x \in \mathbb{R} \quad &; \quad q(x) \vee r(x) \\ \exists x \in \mathbb{Z} \quad &; \quad q(x) \wedge r(x) \\ \exists x > 0 \quad &; \quad (\forall y \in \mathbb{Z}; q(y) \rightarrow r(x)) \\ \sim (\forall x \in \mathbb{Z} \quad &; \quad r(x) \rightarrow q(x)) \end{aligned}$$

Q6. We are given these two relations from $A := \{w, x, y, z\}$ to $B := \{1, 2, 3\}$:

$$R := \{(w,3),(x,1),(w,2),(y,1),(z,2)\} \ , \quad S := \{(x,2),(y,3),(z,2)\}$$

(a) Explain why neither R nor S are functions. [2]

(b) Determine the sets of pairs and the arrow diagrams for $S^{-1} \circ R$ and $S^{-1} \circ S$. Are either of these compound relations reflexive or transitive? [5]

(c) For any relations R and S between the same two sets, $R \cup S$ is also such a relation. Given an appropriate T explain why $T \circ (R \cup S) = (T \circ R) \cup (T \circ S)$. Check this equality for the given R and S and $T = S^{-1}$. [5]

Q7. (a) Prove by contradiction that if $|A \cup B| < |B| + |A|$ then $(A \cap B) \neq \varnothing$. [5]

(b) If $A = B = \varnothing$, why does part (a) still hold? [1]

(c) If $|A| > \frac{|\mathcal{U}|}{2}$ and $|B| \geq \frac{|\mathcal{U}|}{2}$, use (a) to show that $|A \cap B| \geq 1$. [4]

(d) Give an example of three subsets E, F and G of a set of cardinality 12 for which each subset has as large a cardinality as possible but $E \cap F \cap G = \varnothing$. [2]

END OF QUESTION PAPER