Math205 Handout 2: Methods of Proof

We shall show how to go about proving the statement "if an odd integer is multiplied by -1 and that new integer is then added to 26 the result is odd". We first identify the statements $p(x)$ and $q(x)$ in the above statement if it is $p(x) \to q(x)$ and deduce that:

$$\begin{array}{lllll} p(x) & :\equiv & \text{“}x\text{ is odd ”} & \equiv & \text{“}x = 2j+1 \text{ for some } j \in \mathbb{Z}'' \\ q(x) & :\equiv & \text{“}26 - x \text{ is odd”} & \equiv & \text{“}26 - x = 2k+1 \text{ for some } k \in \mathbb{Z}\text{”} \end{array}$$

We are usually either told to use one of the methods below, or we can choose one:

- **Direct:** We suppose that $p(x)$ is true and using what that tells us about x we then apply that to the subject of $q(x)$ in order to try to show that it is true when $p(x)$ is.

  So if $p(x)$ is true then $x = 2j + 1$, and $q(x)$ is about $26 - x$, and putting these two things together

  $$26-x = 26-(2j+1) = 26-2j-1 = 25-2j = 25+2\times(-j) = 1+24+2\times(-j) = 2\times(12-j)+1$$

  Thus we have our statement in the form of $q(x)$ where $k = 12 - j$, and it just remains to establish that this k is an integer. Since j is, multiplying by -1 means $-j$ is still an integer, and then subtracting this from 12, another integer, means that k is an integer as required.

- **Contrapositive:** We can alternatively suppose that $q(x)$ is false and using what that tells us about x we then apply that to the subject of $p(x)$ in order to try to show that it is false when $p(x)$ is. We are thus proving $(\sim q(x)) \to (\sim p(x))$ which we know is logically equivalent to $p(x) \to q(x)$.

  So if $q(x)$ is false we use the fact that, for integers, not being odd is the same as being even, so "$26 - x = 2m; m \in \mathbb{Z}$", and $\sim p(x)$ says that "$x = 2n; n \in \mathbb{Z}$". Simplifying $(\sim q(x))$:

  $$\begin{aligned} 26 - x &= 2m \\ x &= 26 - 2m \\ &= 2 \times (13 - m) \end{aligned}$$

  This equals $2n$ if we take $n = 13 - m$ and so, again, since m is an integer, so is $-m$ and adding 13 to this keeps it an integer, so n is an integer as required.

- **Contradiction:** We now suppose that $p(x)$ is true and also that $q(x)$ is false. We intend to get an impossible situation arising whence we can use the logical equivalence of $(p(x) \wedge (\sim q(x))) \leftrightarrow (\sim T_0)$ and $(p(x) \to q(x)) \leftrightarrow T_0$ to show that $p(x)$ implies $q(x)$ as required.

  As before, if $p(x)$ is true then $x = 2j + 1$, and $(\sim q(x))$ says that $26 - x = 2m$. Combining these two statements to remove x we get:

  $$\begin{aligned} 26 - (2j+1) &= 2m \\ 25 &= 2j + 2m \\ &= 2(j+m) \\ \frac{25}{2} &= j + m \end{aligned}$$

  This statement is our desired contradiction since both j and m are integers and so their sum is an integer, but $\frac{25}{2}$ is certainly not an integer as it is 12.5 in decimal terms and no integer has to be written with a decimal point.

Proof by Induction

- **Induction:** Given a statement $p(n)$ about an integer n we wish to show it is true for all integer values of n at least a and we proceed as follows:
  - *Initial Case:* Show that $p(a)$ is true (optionally also test $p(a+1)$ and $p(a+2)$ to see how the induction will proceed).
  - *Inductive Case:* Assume $p(k)$ is true for some value of $k \geq a$. State one side of $p(k+1)$ in terms of the corresponding side of $p(k)$ and use the assumptions to deduce that the other side of $p(k+1)$ is related in the same way as $p(n)$ was.

For example: $p(n) :\equiv$ "$\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$"

  - *Initial Case:* The first possible value of n is 1, so we consider $p(1) :=$ "$1^2 = 1 = \frac{1\times(1+1)\times(2\times1+1)}{6} = 1$" as required. Similarly, $p(2) :=$ "$1^2+2^2 = 5 = \frac{2\times(2+1)\times(2\times2+1)}{6} = 5$" and $p(3) :=$ "$1^2+2^2+3^2 = 5+3^2 = 14 = \frac{3\times(3+1)\times(2\times3+1)}{6} = 14$".
  - *Inductive Case:* Assume $p(k) :=$ "$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$". Now the left hand side (LHS) of $p(k+1)$ is

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 = \text{LHS}(p(k)) + (k+1)^2.$$

But using the assumption (the inductive hypothesis), we get that

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\frac{k(2k+1)+6(k+1)}{6} \\
&= (k+1)\frac{(2k^2+7k+6)}{6} \\
&= (k+1)\frac{(2k+3)(k+2)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

But this is exactly the statement $p(k+1)$ that we wished to establish!

- **Basic formulae in Sigma Notation:**

$$1+\ldots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad b+\ldots+b = \sum_{i=1}^{n} b = nb \qquad 1+x+\ldots+x^n = \sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$$