Test 3 2018

Q1 $M = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 9 & 6 \\ 0 & -8 & -5 \end{pmatrix}$

(a) $M\underline{v}_1 = \begin{pmatrix} 0+2-2 \\ 0+9-6 \\ 0-8+5 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} = 3 \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ so $\lambda_1 = 3$

b) $\det(M - \lambda I) = \det\left(\begin{pmatrix} -3-\lambda & 2 & 2 \\ -2 & 9-\lambda & 6 \\ 0 & -8 & -5-\lambda \end{pmatrix}\right)$

$= (-3-\lambda) \times \left((9-\lambda)(-5-\lambda) - -8 \times 6\right) - -2 \times (-10-2\lambda - -8\times 2) + 0$

$= -(\lambda+3) \times (\lambda^2 + 5\lambda - 9\lambda - 45 + 48) + 2 \times (-2\lambda + 6)$

$= -(\lambda+3) \times (\lambda^2 - 4\lambda + 3) + 4(-\lambda + 3)$

$= -\lambda^3 + 4\lambda^2 - 3\lambda - 3\lambda^2 + 12\lambda - 9 - 4\lambda + 12$

$= -\lambda^3 + \lambda^2 + 5\lambda + 3$

$\lambda_1 = 3$ is a root so

$$\begin{array}{r} \lambda^2 + 2\lambda + 1 \\ -\lambda + 3 \;\big)\; \overline{-\lambda^3 + \lambda^2 + 5\lambda + 3} \\ -\lambda^3 + 3\lambda^2 \qquad\qquad \\ \hline 2\lambda^2 + 5\lambda \qquad \\ 2\lambda^2 + 6\lambda \qquad \\ \hline -\lambda + 3 \\ -\lambda + 3 \\ \hline 0 \end{array}$$

so $p(\lambda) = -(\lambda - 3)(\lambda^2 + 2\lambda + 1)$
$= -(\lambda - 3)(\lambda + 1)^2$

so $\lambda_2 = \lambda_3 = -1$

$\underline{v}_2$:

$\left(\begin{array}{ccc|c} -2 & 2 & 2 & 0 \\ -2 & 10 & 6 & 0 \\ 0 & -8 & -4 & 0 \end{array}\right)$ $R_1 \times -\frac{1}{2}$, $R_3 \times -\frac{1}{4}$ $\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -5 & -3 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$ Row2 ← Row2 − Row1

$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -4 & -2 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$ Row2 ← Row2 + 2Row3, Row1 ← Row1 + R2 → $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right)$

so
Row1: $x + y = 0$
unpivot: $y = t$
Row 3: $2y + z = 0$

$x = -t$
$y = t$
$z = -2t$

$\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times -t$

check $M\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3-2+4 \\ -2-9+12 \\ 0+8-10 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$

Q2

(a)

$$2A(I+X) = (A^T - I)A$$

A must be n×n to be multipliable like this

× $\frac{1}{2}A^{-1}$ on left (A needs to be invertible)

$$A^{-1}A(I+X) = \frac{1}{2}A^{-1}(A^T - I)A$$

left inverse

$$I + X = \frac{1}{2}\left(A^{-1}A^TA - A^{-1}IA\right)$$

left/right distrib

subtract I

$$X = \frac{1}{2}\left(A^{-1}A^TA - A^{-1}A\right) - I$$

$$= \frac{1}{2}A^{-1}A^TA - \frac{1}{2}I - I = \frac{1}{2}A^{-1}A^TA - \frac{3}{2}I$$

(b)

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \quad A^T = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad A^{-1} = \frac{1}{9-8}\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$$

So

$$X = \frac{1}{2}\left(\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} 9-16 & 6-12 \\ -6+12 & -4+9 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{pmatrix} -7 & -6 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} -21-12-3 & -28-18 \\ 18+10 & 24+15-3 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -36 & -46 \\ 28 & 36 \end{pmatrix} = \begin{pmatrix} -18 & -23 \\ 14 & 18 \end{pmatrix}$$

all integers