Math1204 Handout 5: Lines, Planes, etc.

We can use matrices to model sets of points in spaces of any dimension. We use $n \times 1$ column vectors to give the position in the space and define the dot product of two vectors $\underline{v}$ and $\underline{w}$ as $\underline{v} \circ \underline{w} := \underline{v}^T \underline{w}$, using matrix transposition and multiplication.

- The familiar x-y plane is the 2-dimensional vector space $\mathbb{R}^2$, where x and y can be any real numbers, usually x is the horizontal distance and y is the vertical distance. The standard equation of a line $y = mx + b$ can be represented in vector format as follows, substituting $x = \alpha$ as it is any real number:

$$\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx+b \end{pmatrix} = \begin{pmatrix} 1 \\ m \end{pmatrix} \times \alpha + \begin{pmatrix} 0 \\ b \end{pmatrix} = \underline{w} \times \alpha + \underline{u}$$

- In general n dimensional space $\mathbb{R}^n$ we have a similar format for a line: $\underline{v} = \underline{w} \times \alpha + \underline{u}$, but now $\underline{u}$ (a point on the line) and $\underline{w}$ (the direction of the line, which can't be $\underline{0}$) have n numbers in them and $\underline{v}$ is our general point on the line.

- In a similar way we can describe a 2-dimensional set of points, using two directions. Such a set is called a plane, so long as the directions are different so it doesn't collapse to just give us a line. The general equation for a plane in n dimensions uses the directions $\underline{w}_1$ and $\underline{w}_2$ or (in 3d) $\underline{n}$ the normal vector:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{w}_1 \times \alpha_1 + \underline{w}_2 \times \alpha_2 + \underline{u} \quad , \quad \text{in 3d only}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \circ \underline{n} = t \quad (\underline{w}_1 \circ \underline{n} = 0, \underline{w}_2 \circ \underline{n} = 0)$$

- The condition for two directions to be different is just that $\underline{w}_1$ and $\underline{w}_2$ are not multiples of each other, but we can generalise this idea to give us the concept of *independent vectors*. We want to be sure that the only solution to the following equation is the trivial one, that is, if

$$\sum_{j=1}^{k} (\underline{w}_j \times \alpha_j) = \underline{w}_1 \times \alpha_1 + \underline{w}_2 \times \alpha_2 + \ldots + \underline{w}_k \times \alpha_k = \underline{0} \quad \text{or} \quad \left(\begin{array}{cccc|c} & & & & 0 \\ \underline{w}_1 & \underline{w}_2 & \ldots & \underline{w}_k & 0 \\ & & & & 0 \end{array} \right)$$

  then we must have $\alpha_1 = 0$, $\alpha_2 = 0$, $\ldots$, $\alpha_k = 0$. We can represent this as a homogeneous system of equations that only has the trivial solution, or equivalently, the rank of the matrix of vectors is k.

- A general space of dimension k in $\mathbb{R}^n$ is made from k independent direction vectors ($k \leq n$).

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{w}_1 \times \alpha_1 + \underline{w}_2 \times \alpha_2 + \ldots + \underline{w}_k \times \alpha_k + \underline{u}$$

- Just as in $\mathbb{R}^2$ we can have parallel lines, we can have parallel planes, or lines parallel to planes, but there is a useful general rule that can indicate what kind of set of points is most likely to be in two sets, using dimensions. If we are in n-dimensional space ($\mathbb{R}^n$) and want to know what the intersection of m and p dimensional spaces should be, then we just consider $s := m + p - n$.

  If s is negative then probably they don't intersect, and if $s = 0$ then it should be a point, $s = 1$ should be a line, $s = 2$ a plane. For instance, in 3d space two planes will usually meet in a line since $s = 2 + 2 - 3 = 1$. In 4d two planes will usually only meet at a point as $s = 2 + 2 - 4 = 0$ and a plane and a line probably wont meet since $s = 1 + 2 - 4 = -1$. Also, s can't be greater than either m or p.