Q1 (a) $5(A X A^{-1} - A) = AB - 3I$ $\quad \times \frac{1}{5}$

$$A X A^{-1} - A = \frac{1}{5}(AB - 3I)$$
$$= \frac{1}{5}AB - \frac{3}{5}I$$
$$+A$$
$$A X A^{-1} = \frac{1}{5}AB - \frac{3}{5}I + A$$
$\times A$ on right
$$(A X A^{-1})A = \left(\frac{1}{5}AB - \frac{3}{5}I + A\right)A$$
Assoc/R distrib
$$A X (A^{-1}A) = \frac{1}{5}ABA - \frac{3}{5}IA + AA$$
Left ident/inverse
$$AX = AXI = \frac{1}{5}ABA - \frac{3}{5}A + A^2$$
$\times A^{-1}$ on left
$$A^{-1}(AX) = A^{-1}\left(\frac{1}{5}ABA - \frac{3}{5}A + A^2\right)$$
$$X = IX = \frac{1}{5}A^{-1}(ABA) - \frac{3}{5}A^{-1}A + A^{-1}AA$$
Assoc, left ident/inverse
$$X = \frac{1}{5}(A^{-1}A)BA - \frac{3}{5}I + IA = \frac{1}{5}BA - \frac{3}{5}I + A$$

(b) Since A^{-1} exists (A is non singular) A must be $n \times n$.
As $X = \sim + A$, X and A must be the same size
BA is same size as A so BA is $n \times n$, and B must be $m \times n$ to multiply but then BA is $m \times n$ so B is $n \times n$ as $m = n$.

(c) $$\frac{1}{5}(BA) - \frac{3}{5}I + A = \frac{1}{5}\begin{pmatrix} 4 & -5 \\ -13 & 6 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 11 & 8 \end{pmatrix} - \frac{3}{5}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 7 & 5 \\ 11 & 8 \end{pmatrix}$$
$$= \frac{1}{5}\begin{pmatrix} 28-55 & 20-40 \\ -91+66 & -65+48 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 35 & 25 \\ 55 & 40 \end{pmatrix}$$
$$= \frac{1}{5}\begin{pmatrix} -27-3+35 & -20+25 \\ -25+55 & -17-3+40 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 & 5 \\ 30 & 20 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 \\ 6 & 4 \end{pmatrix}$$

and all are positive and below 9

Q2 (a) $M\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}-32+40-10\\15-18+5\\45-60+13\end{pmatrix} = \begin{pmatrix}-2\\2\\-2\end{pmatrix} = -2\times\begin{pmatrix}+1\\-1\\1\end{pmatrix}$ so $\lambda_1=-2$

(b) $M-\lambda I = \begin{pmatrix}-32-\lambda & -40 & -10\\15 & 18-\lambda & 5\\45 & 60 & 13-\lambda\end{pmatrix}$

$\det(M-\lambda I) \overset{C_1\leftarrow C_1-3C_3}{=} \det\left(\begin{pmatrix}-2-\lambda & -40 & -10\\0 & 18-\lambda & 5\\6+3\lambda & 60 & 13-\lambda\end{pmatrix}\right) \overset{R_3\leftarrow R_3+3R_1}{=} \det\left(\begin{pmatrix}-2-\lambda & -40 & -10\\0 & 18-\lambda & 5\\0 & -60 & -17-\lambda\end{pmatrix}\right)$

(c) so $\det(M-\lambda I) = (-2-\lambda)\times\det\left(\begin{pmatrix}18-\lambda & 5\\-60 & -17-\lambda\end{pmatrix}\right)$

expand column 1 $=(-2-\lambda)(\lambda^2+17\lambda-18\lambda-306+300)$

$=(-2-\lambda)(\lambda^2-\lambda-6)$

$=(-2-\lambda)(\lambda-3)(\lambda+2)$ $\quad(\lambda-3)(\lambda+2)=\lambda^2-3\lambda+2\lambda-6$ ✓

so $\underbrace{\lambda_1=\lambda_2=-2}_{\text{multiplicity }2}$ $\lambda_3=3$

(d) $M-3I = \begin{pmatrix}-35 & -40 & -10\\15 & 15 & 5\\45 & 60 & 10\end{pmatrix}$

solving homog: $R_1\leftarrow R_1+2R_2$ $R_3\leftarrow R_3-2R_2$

$\left(\begin{array}{ccc|c}-5 & -10 & 0 & 0\\15 & 15 & 5 & 0\\15 & 30 & 0 & 0\end{array}\right)$

$R_3\leftarrow R_3+3R_1$ $R_2\leftarrow R_2+3R_1$

$\left(\begin{array}{ccc|c}-5 & -10 & 0 & 0\\0 & -15 & 5 & 0\\0 & 0 & 0 & 0\end{array}\right)$

so $5x=-10y$ $5z=15y$

$y=t$ $x=-2t$ $z=3t$

$\underline{v}_3=\begin{pmatrix}-2\\1\\3\end{pmatrix}$

(e) $M+2I = \begin{pmatrix}-30 & -40 & -10\\15 & 20 & 5\\45 & 60 & 15\end{pmatrix}$

$R_1\leftarrow R_1+2R_2$ $R_3\leftarrow R_3-3R_2$

$\left(\begin{array}{ccc|c}0 & 0 & 0 & 0\\15 & 20 & 5 & 0\\0 & 0 & 0 & 0\end{array}\right)$

$R_2\leftarrow R_2\times\frac{1}{5}$

$z=-3x-4y$ $\quad x=p$ $y=q$

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\underline{\underline{\begin{pmatrix}1\\0\\-3\end{pmatrix}}}p+\underline{\underline{\begin{pmatrix}0\\1\\-4\end{pmatrix}}}q$ Two eigenvectors

If we take $p=1, q=-1$ then $\begin{pmatrix}x\\y\\z\end{pmatrix}=\underline{\underline{\begin{pmatrix}1\\-1\\1\end{pmatrix}}}$

Q3 (a) $\det(H-\lambda I) = \left(\frac{17}{2}-\lambda\right)(-7-\lambda) - 10\times -6$

$$= \lambda^2 + 7\lambda - \frac{17}{2}\lambda - \frac{119}{2} + \frac{120}{2}$$

$$= \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$$

$$= \left(\lambda - \frac{1}{2}\right)(\lambda - 1) = \lambda^2 - \frac{1}{2}\lambda - \frac{2}{2}\lambda + \frac{1}{2}$$

$\underline{v}_1$ $\left(\begin{array}{cc|c} \frac{17}{2}-\frac{1}{2} & -6 & 0 \\ 10 & -7-\frac{1}{2} & 0 \end{array}\right)$ $\left(\begin{array}{cc|c} 8 & -6 & 0 \\ 10 & -\frac{15}{2} & 0 \end{array}\right)$ $R_2 \leftarrow R_2 - \frac{10}{8}R_1$ $\left(\begin{array}{cc|c} 8 & -6 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\underline{v}_1 = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$ or $\underline{\underline{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}}$

$\underline{v}_2$ $\left(\begin{array}{cc|c} \frac{17}{2}-1 & -6 & 0 \\ 10 & -7-1 & 0 \end{array}\right)$ $\left(\begin{array}{cc|c} \frac{15}{2} & -6 & 0 \\ 10 & -8 & 0 \end{array}\right)$ $R_2 \leftarrow R_2 - \frac{4}{3}R_1$ $\left(\begin{array}{cc|c} \frac{15}{2} & -6 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\underline{v}_2 = \begin{pmatrix} 6 \\ \frac{15}{2} \end{pmatrix}$ or $\begin{pmatrix} 12 \\ 15 \end{pmatrix}$ or $\underline{\underline{\begin{pmatrix} 4 \\ 5 \end{pmatrix}}}$

(b) $H^k = PD^kP^{-1} = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} \left(\frac{1}{2}\right)^k & 0 \\ 0 & 1^k \end{pmatrix} \begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}$ $\qquad \frac{1}{15-16}\begin{pmatrix} 5 & -4 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix}$

$$= \begin{pmatrix} 3\times\left(\frac{1}{2}\right)^k & 4 \\ 4\times\left(\frac{1}{2}\right)^k & 5 \end{pmatrix} \begin{pmatrix} -5 & 4 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 16 - 15\times\left(\frac{1}{2}\right)^k & 12\times\left(\left(\frac{1}{2}\right)^k - 1\right) \\ 20\left(1-\left(\frac{1}{2}\right)^k\right) & 16\left(\frac{1}{2}\right)^k \cancel{-} 15 \end{pmatrix}$$

(c) $\left(\frac{1}{2}\right)^7 = \frac{1}{128}$ so $H^k \begin{pmatrix} 192 \\ 160 \end{pmatrix} = \frac{1}{128}\begin{pmatrix} 16\times 128 - 15 & 12\times -127 \\ 20\times +127 & 16 - 15\times 128 \end{pmatrix} \begin{pmatrix} 192 \\ 160 \end{pmatrix}$

$$= \frac{1}{128}\begin{pmatrix} 2033 & -1524 \\ 2540 & -1904 \end{pmatrix}\begin{pmatrix} 192 \\ 160 \end{pmatrix}$$

Note $\frac{1144.5}{1430} = 0.80035 \approx \frac{4}{5}$ $\qquad = \frac{1}{128}\begin{pmatrix} 146496 \\ 183040 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1144.5 \\ 1430 \end{pmatrix}}}$

For $\to 0$ we need a zero coeff for our dominant eigenvalue $1 > \left|\frac{1}{2}\right| = \frac{1}{2}$

so the ratio must be $\frac{3}{4} = \frac{192}{c}$ so $c = \frac{4\times 192}{3} = 256$

Q4

(a)
$$\operatorname{adj}(J) = \begin{pmatrix} \det\begin{pmatrix}4 & s\\ -2 & 2\end{pmatrix} & -\det\begin{pmatrix}3 & s\\ 5 & 2\end{pmatrix} & +\det\begin{pmatrix}3 & 4\\ 5 & -2\end{pmatrix} \\ -\det\begin{pmatrix}-3 & 1\\ -2 & 2\end{pmatrix} & +\det\begin{pmatrix}1 & 1\\ 5 & 2\end{pmatrix} & -\det\begin{pmatrix}1 & -3\\ 5 & -2\end{pmatrix} \\ \det\begin{pmatrix}-3 & 1\\ 4 & s\end{pmatrix} & -\det\begin{pmatrix}1 & 1\\ 3 & s\end{pmatrix} & \det\begin{pmatrix}1 & -3\\ 3 & 4\end{pmatrix}\end{pmatrix}^T$$

$$= \begin{pmatrix} 8+2s & -(6-5s) & -6-20 \\ -(-6+2) & 2-5 & -(-2+15) \\ -3s-4 & -(s-3) & \cancel{4}-9 \end{pmatrix}^T = \begin{pmatrix} 8+2s & 5s-6 & -26 \\ 4 & -3 & -13 \\ -3s-4 & 3-s & 13 \end{pmatrix}^T$$

$$= \begin{pmatrix} 8+2s & 4 & -3s-4 \\ 5s-6 & -3 & 3-s \\ -26 & -13 & 13 \end{pmatrix}$$

$$J \times \operatorname{adj}(J) = \begin{pmatrix} 8+2s-15s+18s-26 & 4+9-13 & -3s-4-9+3s+13 \\ 24+6s+20s-24-26s & 12-12-13s & -9s-12+12-4s+13s \\ 40+10s-10s+12-52 & 20+6-26 & -15s-20-6+2s+26 \end{pmatrix}$$

$$= \begin{pmatrix} -13s & 0 & 0 \\ 0 & -13s & 0 \\ 0 & 0 & -13s \end{pmatrix}$$

so $\det(J) = -13s$

(b) (i)
$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 3 & 4 & 8 & 1 \\ 5 & -2 & 2 & 1 \end{array}\right) \begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ R_3 \leftarrow R_3 - 5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 0 & 13 & 5 & 1 \\ 0 & 13 & -3 & 1 \end{array}\right) \begin{array}{l} R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 + \frac{3}{13} R_3 \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \cancel{4/13} & 3/13 \\ 0 & 0 & 1 & 0 \\ 0 & 13 & 0 & 1 \end{array}\right) \xleftarrow{} \begin{array}{l} R_2 \leftarrow R_2 \times \frac{1}{8} \\ R_1 \leftarrow R_1 - \frac{4}{13} \text{new}R_2 \\ R_3 \leftarrow R_3 + 3\,\text{new}R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 4/13 & 3/13 \\ 0 & 0 & 8 & 0 \\ 0 & 13 & -3 & 1 \end{array}\right)$$

so
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3/13 \\ 1/13 \\ 0 \end{pmatrix}$$

check
$$\frac{1}{13} J \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 3-3 \\ 9+4 \\ 15-2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \checkmark$$

ii) Note that with row ops s would change to $s-3$ and then back to s giving row $(0\ 0\ s : 0)$. Thus $sz = 0$ which means $z = 0$ (unless $s = 0$) (which is also when $\det J = 0$)

iii) In this case we have $x = -\frac{4}{13} z$ $\quad y = \frac{3}{13} z$ $\quad z = \frac{13}{13} z$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 13 \end{pmatrix}$$
for homog solutions