Existence and nonexistence of positive radial solutions for a class of semilinear elliptic system

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1 Introduction

The main purpose of this article is to establish some existence and nonexistence results of

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positive radial solutions to the following semilinear elliptic system:

$$\triangle u + f(u,v) = 0 \quad \text{and} \quad \triangle v + g(u,v) = 0 \quad \text{in } R^n, n \geq 3. \tag{1}$$

For special f, g we will give conditions to guarantee that all radial solutions of (1) are positive on R^n, and that the positive radial solutions exist or do not exist on R^n or on a certain ball $B(R)$.

Existence and nonexistence of positive classical solutions for the scalar equation

$$\triangle u = f(x,u) \quad in \ R^n,$$

has been investigated extensively. There has been tremendous amount of references in this area, we shall not attempt to mention all of them. We list here, for example, [GS], [N], [BN], [R] and so on, to just name a few.

In [CK], symmetry results for elliptic system of Liouville type in R^2 were given. For the corresponding theory for elliptic systems in R^n ($n \geq 3$), not much is known as far as the existence and nonexistence results are concerned, except for some special f and g; or at least no complete picture has been given yet. David [D] considered the problem (1) with $f(u,v) = \partial F(u,v)/\partial u$ and $g(u,v) = \partial F(u,v)/\partial v$ for some function F. His aim was to give sufficient conditions on F to guarantee the existence of infinitely many radial solutions. Mitidieri [M] discussed nonexistence of positive solutions of the problem (1). Two of his main theorems are the following nonexistence results:

(i) Suppose that $f(u,v) \geq c_1 u^k + c_2 v^p$ and $g(u,v) \geq c_3 u^q + c_4 v^t$ for all $u, v \in [0, \infty)$, where $f(0,0) = g(0,0) = 0$, $c_i > 0$ and $k, t, p, q > 1$. If

$$\frac{n-2}{2} \leq \max\left\{\frac{1}{k-1}, \frac{p+1}{pq-1}, \frac{1}{t-1}, \frac{q+1}{pq-1}\right\},$$

then the problem (1) has no positive solutions of $C^2(R^n)$.

(ii) Suppose that $f(u,v) = u^k v^p$ and $g(u,v) = u^q v^t$ with $0 \leq k, t \leq 1$ and $p, q > 1$. If

$$\frac{n}{2} \leq \min\left\{\frac{(p+1)(q+1) + k(q+1) - tk}{pq - (1-t)(1-k)}, \frac{(p+1)(q+1) + k(q+1) + t(p+1)}{pq + k(q+1) + t(p+1) - 1}\right\},$$

then the problem (1) has no positive solutions of $C^2(R^n)$.

When $f(u,v) = v^p$ and $g(u,v) = u^q$, where $p, q > 0$ and $n \geq 3$, Serrin and Zou [SZ] proved that the hyperbola

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-2}{n}$$

is the dividing curve for existence and nonexistence on the $pq-$plane in several important cases.

As corollaries of their main results, Serrin and Zou [SZ] established for the case $f(u,v) = v^p$ and $g(u,v) = u^q$ the following (see [SZ] for details):

(I) Suppose $n = 3$. Then the hyperbola defined above is the dividing curve for positive solutions of problem (1) with algebraic growth at infinity.

(II)The hyperbola above is the dividing curve for positive radial solutions of problem (1).

In [Z], Zou considered the problem (1) for $f(u,v) = -v + v^p$ and $g(u,v) = -u + u^q$. He proved, among other things, that in this case problem (1) admits ground states if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}.$$

We restrict our study in this paper to the positive radial solutions. For radial solutions, the above equations will lead to the study of the following elliptic system:

$$u''(r) + \frac{n-1}{r}u'(r) + f(u(r), v(r)) = 0, \tag{2}$$

$$v''(r) + \frac{n-1}{r}v'(r) + g(u(r), v(r)) = 0, \tag{3}$$

$$u(0) > 0, v(0) > 0, u'(0) = v'(0) = 0, \tag{4}$$

in R^n with $n \geq 3$, where we assume

$$f(u,v) \geq au^k v^p \quad \text{and} \quad g(u,v) \geq bu^q v^t, \tag{5}$$

for all $u, v \in (0, \infty)$ with $k,\ t > 1,\ \ p,\ q \geq 0$ and a, b are positive constants.

Our theorems for nonexistence extend the exponents k, p, q, t to $k, t \geq 1$ and $p, q \geq 0$ compared with those derived in [M]. The methods we employ in this paper appear to be quite different than those in [M], and our theorems include both nonexistence and existence of positive radial solutions. Moreover, our theorems are much simply stated and can be more easily verified for the exponents k, p, q, t.

The first main result we prove here is **Theorem 1** in section 2. This theorem states:

Suppose that (5) holds. If $n > (n-2)\min(k+p,\ q+t)$, then (2) - (4) has no positive solutions in $(0, \infty)$.

For special $f,\ g$ we will discuss the following problems

$$u''(r) + \frac{n-1}{r}u'(r) + au^k(r)v^p(r) = 0, \tag{6}$$

$$v''(r) + \frac{n-1}{r}v'(r) + bu^q(r)v^t(r) = 0, \tag{7}$$

Our second main result in this paper is **Theorem 2** in section 2:

If $n+2 \geq (n-2)\min(k+2p,\ t+2q)$, then (6), (7) and (4) has no positive solutions in $(0, \infty)$.

If we restrict the exponents k, p, q, t such that $q = k-1$ and $t = p+1$, we will be able to prove more precise results. In this case, the main theorems we derive are stated as follows:

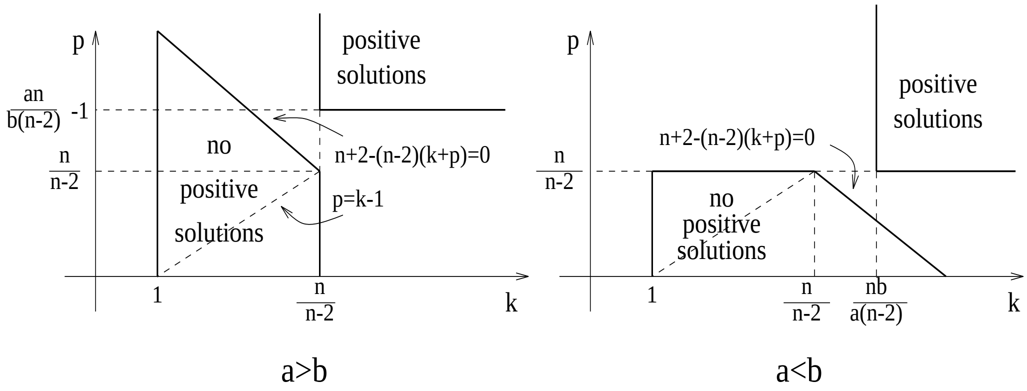
1) (**Theorem 4** in section 3) Suppose that $k(n-2) \geq n$ and $b(n-2)(p+1) \geq na$ if $a \geq b$ and that $p(n-2) \geq 2$ and $a(n-2)k \geq nb$ if $b > a$. Then, for any positive initial values $u(0)$ and $v(0)$, the solutions $u(r)$ and $v(r)$ of (21) - (23) are positive in $(0, \infty)$.

2) (**Theorem 5** in section 3) If $(n-2)k < n$ and $(n-2)p < 2$, then there are no positive solutions in $(0, \infty)$ for problem (21) - (23).

3) (**Theorem 6** in section 3) Let $l = n + 2 - (n-2)(k+p) > 0$. Suppose that $k - 1 > p$ if $b \geq a$ and that $k - 1 < p$ if $a \geq b$. Then,there are no positive solutions in $(0, \infty)$ for problem (21) - (23).

As we have noticed above in the study of the existence and nonexistence results for global positive solutions, though both the exponents k, p, q, t and the coefficient constants a, b of the nonlinear terms play crucial roles in some cases (see Theorems 4 and 6), a, b seem to be irrelevant in some other cases (see Theorems 1, 2 and 5). These indicate the very subtle and delicate nature of the problem under study here.

These three theorems above (Theorems 4, 5, and 6) can be outlined in the following figue:



We will also mention below that for the local Dirichlet boundary problem, the situation turns out to be quite surprising. We just present one result here to distinguish the difference.

We consider the elliptic system on a certain ball $B = B(0, R)$, then we need to consider the following Dirichlet boundary problem:

$$\triangle u + au^k v^p = 0 \ and \ \triangle v + bu^{k-1}v^{p+1} = 0 \tag{8}$$

with the following boundary condition:

$$u|_{\partial B} = 0, v|_{\partial B} = 0 \tag{9}$$

for $B = B(0, R) \subset R^n, n \geq 3$.

We find the following local nonexistence result when $q = k-1$ and $t = p+1$ rather interesting:

(**Theorem 3** in section 3) If $a \neq b$, then the Dirichlet boundary problem (8) - (9) has no positive radial solutions.

As shown in the above theorem, the coefficient constants a, b of the nonlinear terms seem to dominate to some extent the nonexistence of the positive radial solutions on any bounded ball regardless of the eligible exponents k, p, q, t

Mitidieri [M] also discussed the similar problem. He supposed that, for example, $f(u, v) = u^k v^p$ and $g(u, v) = u^q v^t$ with $k, p, q, t > 1$ and

$$\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}.$$

Then the problem (1) together with (9) has no positive solutions.

In [Z], Zou established some existence results on balls for a different type of systems for $f = -v + v^p$ and $g = -u + u^q$. He proved that this problem admits a positive solution (not necessarily radial) with homogeneous Dirichlet boundary condition if and only if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}.$$

In this paper we suppose that c is a generic constant, not necessarily the same in different formulas.

2 Nonexistence results with four indices

In this section, we will prove the nonexistence theorems stated in the last section. We first observe that equations (2) - (4) can be rewritten as

$$\left(r^{n-1}u'\right)' = -r^{n-1}f(u,v), \quad \left(r^{n-1}v'\right)' = -r^{n-1}g(u,v), \tag{10}$$

where $f(u,v) \geq au^k v^p$ and $g(u,v) \geq bu^q v^t$. Integrating these equalities from 0 to r yields

$$r^{n-1}u'(r) = -\int_0^r f(u(s),v(s))s^{n-1}ds, \quad r^{n-1}v'(r) = -\int_0^r g(u(s),v(s))s^{n-1}ds,$$

so that

$$u'(r) = -\frac{1}{r^{n-1}}\int_0^r f(u(s),v(s))s^{n-1}ds, \quad v'(r) = -\frac{1}{r^{n-1}}\int_0^r g(u(s),v(s))s^{n-1}ds. \tag{11}$$

We first obtain the following decay properties of u and v.

Lemma 1 *Suppose that condition (5) holds. If $u(r)$ and $v(r)$ are positive solutions of (2)-(4) in $(0,\infty)$, then for $r > 0$,*

$$u^{k-1}(r)v^p(r) \leq \frac{2n}{a(k-1)}r^{-2}, \quad and \quad u^q(r)v^{t-1}(r) \leq \frac{2n}{b(t-1)}r^{-2}. \tag{12}$$

Proof. From (11), $u(r)$ and $v(r)$ are decreasing functions in $(0,\infty)$, which implies that

$$u'(r) \le -\frac{a}{r^{n-1}}\int_0^r u^k(s)v^p(s)s^{n-1}ds \le -\frac{a}{r^{n-1}}u^k(r)v^p(r)\int_0^r s^{n-1}ds = -\frac{a}{n}ru^k(r)v^p(r), \tag{13}$$

$$v'(r) \le -\frac{b}{r^{n-1}}u^q(r)v^t(r)\int_0^r s^{n-1}ds = -\frac{b}{n}ru^q(r)v^t(r). \tag{14}$$

Using these two inequalities, we get

$$\frac{d}{dr}\left[u^{k-1}(r)v^p(r)\right] = (k-1)\,u^{k-2}(r)v^p(r)u'(r) + pu^{k-1}(r)v^{p-1}(r)v'(r)$$

$$\le -(k-1)u^{k-2}(r)v^p(r)\cdot\frac{a}{n}ru^k(r)v^p(r) = -\frac{a(k-1)r}{n}\left[u^{k-1}(r)v^p(r)\right]^2.$$

Solving this inequality, we have

$$\frac{1}{u^{k-1}(0)v^p(0)} - \frac{1}{u^{k-1}(r)v^p(r)} \le -\frac{a(k-1)}{2n}r^2,$$

or

$$u^{k-1}(r)v^p(r) \le \frac{1}{\frac{1}{u^{k-1}(0)v^p(0)} + \frac{a(k-1)}{2n}r^2} \le \frac{2n}{a(k-1)}r^{-2}.$$

Similarly, we can prove the second inequality of (12). This completes the proof of Lemma 1. We now derive the following

Lemma 2 *If $u(r)$ and $v(r)$ are positive solutions of (2) - (4) in $(0,\infty)$, then*

$$ru'(r) + (n-2)u(r) \ge 0, \quad and \quad rv'(r) + (n-2)v(r) \ge 0. \tag{15}$$

Proof. Suppose that the first inequality of (15) was not true, then there would exist an $r_0 > 0$ such that $h(r_0) = r_0u'(r_0) + (n-2)u(r_0) < 0$. Rewrite (2) as

$$(ru'(r))' + (n-2)u'(r) = -rf((u(r),v(r)) < 0,$$

which implies that

$$ru'(r) + (n-2)u(r) < h(r_0) \quad \text{for} \quad r > r_0,$$

or

$$u'(r) < h(r_0)\frac{1}{r}, \tag{16}$$

Integrating (16) from r_0 to r, we obtain

$$u(r) - u(r_0) < h(r_0) ln \frac{r}{r_0} \to -\infty \quad \text{as} \quad r \to \infty,$$

which is a contradiction. Similarly, the second inequality of (15) holds.

Theorem 1 *If $n > (n-2)$ min $(k+p,\ q+t)$, then the problem (2)-(4) has no positive solutions in $(0, \infty)$.*

Proof. Suppose otherwise, $u(r)$ and $v(r)$ are the positive solutions of (2)-(4) in $(0, \infty)$. By Lemma 2,

$$(r^{n-2}u(r))' = r^{n-3}[ru'(r) + (n-2)u(r)] > 0,$$

we have

$$u(r) \geq c\, r^{-(n-2)} \quad \text{and} \quad v(r) \geq c\, r^{-(n-2)} \quad \text{for } r \geq 1. \tag{17}$$

By Lemma 1 and (17),

$$c\, r^{-2} \geq u^{k-1}(r)v^p(r) \geq c^{k-1+p} r^{-(n-2)(k-1+p)} = c^{k-1+p} r^{-2+n-(n-2)(k+p)},$$

and

$$c\, r^{-2} \geq u^{q-1}(r)v^t(r) \geq c^{q-1+t} r^{-2+n-(n-2)(q+t)},$$

which is a contradiction for large r. Hence (2)-(4) has no positive solutions in $(0, \infty)$, and thus the theorem follows.

To prove the next theorem, we will employ Pokhozhaev's second identity (see [Po]). This identity has also been used by the authors in [CL] in the study of asymptotic behavior of oscillating solutions to a class of semilinear equations. This identity reads as

$$\begin{aligned} & r^{k+1}[\frac{1}{2}u'^2(r) + F(r, u(r))] + \alpha r^k u'(r)u(r) + \frac{\alpha}{2}(n-1-k)r^{k-1}u^2(r) \\ & + (n-1-\frac{k+1}{2}-\alpha)\int_0^r u'^2(s)s^k ds - \alpha\frac{(n-1-k)(k-1)}{2}\int_0^r u^2(s)s^{k-2}ds \\ & + \int_0^r [\alpha u(s)f(s, u(s)) - (k+1)F(s, u(s)) - sF_s(s, u(s))]s^k ds = 0 \end{aligned} \tag{18}$$

for $k>1$, where

$$F(r,u)=\int_0^u f(s,v)dv$$

and $u(r)$ is a solution of

$$u''(r)+\frac{n-1}{r}u'(r)+f(r,u)=0.$$

Theorem 2 *Suppose that* $f(u,v)=au^kv^p$ *and* $g(u,v)=bu^qv^t$ *with* $k,t>1$ *and* $p,q\geq 0$. *If* $n+2\geq(n-2)\ min\ (k+2p,\ t+2q)$, *then the problem (2)-(4) has no positive solutions in* $(0,\infty)$.

Proof. Without loss of generality, we assume $k+2p\leq t+2q$. Suppose that our conclusion is not true, and $u(r),\ v(r)$ are the positive solutions of (2)-(4) in $(0,\infty)$. From (12) and (17),

$$c^p\ r^{-(n-2)p}u^{k-1}(r)\leq u^{k-1}(r)v^p(r)\leq c\ r^{-2},$$

which implies that

$$u(r)\leq c\ r^{-\frac{2-(n-2)p}{k-1}}. \tag{19}$$

Using Pokhozhaev's second identity (18) for the equation (6) with $\alpha=(n-2)/2,\ k=n-1$, we have

$$\begin{aligned}&r^n\Big[\frac{1}{2}u'^2(r)+\frac{a}{k+1}u^{k+1}(r)v^p(r)\Big]+\frac{n-2}{2}r^{n-1}u'(r)u(r)\\&=a\int_0^r\Bigg[\Bigg(\frac{n}{k+1}-\frac{n-2}{2}\Bigg)u^{k+1}(s)v^p(s)+\frac{p}{k+1}u^{k+1}(s)v^{p-1}(s)v'(s)s\Bigg]s^{n-1}ds,\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{2}r^{n-1}u'(r)\Big[ru'(r)+(n-2)u(r)\Big]+\frac{a}{k+1}r^nu^{k+1}(r)v^p(r)\\&=\frac{ap}{k+1}\int_0^r u^{k+1}(s)v^{p-1}(s)\Big[sv'(s)+(n-2)v(s)\Big]s^{n-1}ds\\&+\frac{n+2-(n-2)(k+2p)}{2(k+1)}\int_0^r u^{k+1}(s)v^p(s)s^{n-1}ds.\end{aligned} \tag{20}$$

It follows from (17) and (19) that

$$\begin{aligned}&r^nu^{k+1}(r)v^p(r)\leq c\ r^n[u^{k-1}(r)v^p(r)]\cdot u^2(r)\leq c\ r^{n-2-\frac{4-2(n-2)p}{k-1}}\\&=c\ r^{-\frac{n+2-(n-2)(k+2p)}{k-1}}\to 0\quad \text{as}\quad r\to\infty,\end{aligned}$$

Thus, by Lemma 2, the left side of (20) is less than any $\epsilon > 0$ for large r, but the right side is greater than a positive number. This is a contradiction. Thus our conclusion follows.

3 Existence and nonexistence results with two indices

In this section, we discuss the special cases:

$$u''(r) + \frac{n-1}{r}u'(r) + au^k(r)v^p(r) = 0, \tag{21}$$

$$v''(r) + \frac{n-1}{r}v'(r) + bu^{k-1}(r)v^{p+1}(r) = 0, \tag{22}$$

$$u(0) > 0, v(0) > 0, u'(0) = v'(0) = 0, \tag{23}$$

in R^n with $n \geq 3$, where $k > 1, p \geq 0$ and a, b are positive constants.

We first prove the following comparison lemma between the solutions u and v. This lemma will be used in the proof of Theorem 6 below.

Lemma 3 *Suppose that $u(r)$ and $v(r)$ are any pair of positive solutions of (21) - (23) in $(0, R)$, where $R \leq \infty$. Then*

$$\frac{u(r)}{v(r)} = c_0 + \int_0^r \frac{b-a}{\tau^{n-1}v^2(\tau)} \int_0^\tau u^k(s)v^{p+1}(s)s^{n-1}dsd\tau, \tag{24}$$

and

$$\frac{v(r)}{u(r)} = c_0^{-1} + \int_0^r \frac{a-b}{\tau^{n-1}u^2(\tau)} \int_0^\tau u^k(s)v^{p+1}(s)s^{n-1}dsd\tau, \tag{25}$$

for $0 < r \leq R$, where $c_0 = u(0)/v(0)$.

Proof. Multiplying the first equation of (10) by v and the second one by u, substracting the results, and integrating it from 0 to r gives

$$\int_0^r \Big[(s^{n-1}u'(s))'v(s) - (s^{n-1}v'(s))'u(s)\Big]ds = (b-a)\int_0^r u^k(s)v^{p+1}(s)s^{n-1}ds.$$

Integrating the left side by parts yields

$$r^{n-1}u'(r)v(r) - r^{n-1}v'(r)u(r) = (b-a)\int_0^r u^k(s)v^{p+1}(s)s^{n-1}ds.$$

Then

$$\left(\frac{u(r)}{v(r)}\right)' = \frac{u'(r)v(r) - v'(r)u(r)}{v^2(r)} = \frac{b-a}{r^{n-1}v^2(r)}\int_0^r u^k(s)v^{p+1}(s)s^{n-1}ds.$$

Hence, (24) holds. Similarly, (25) holds.

We now derive an identity which is very useful in the proofs of Theorems 4 and 5.

Lemma 4 *Suppose that $u(r)$ and $v(r)$ are any pair of positive solutions of (21) - (23) in $(0,R)$, where $R \leq \infty$. Let*

$$J(r) = r^n u'(r)v'(r) + (n-2)r^{n-1}\left[\alpha u(r)v'(r) + (1-\alpha)u'(r)v(r)\right] + \beta r^n u^k(r)v^{p+1}(r). \tag{26}$$

Then, for any α, β,

$$J(r) = \left[n\beta - a(n-2) - \alpha(n-2)\left(b-a\right)\right]\int_0^r u^k(s)v^{p+1}(s)s^{n-1}ds$$

$$+\left[\beta(p+1) - a\right]\int_0^r u^k(s)v^p(s)v'(s)s^n ds + \left(\beta k - b\right)\int_0^r u^{k-1}(s)v^{p+1}(s)u'(s)s^n ds. \tag{27}$$

Proof. Differentiating (26) and using the equations (21) and (22) gives

$$\begin{aligned}
\frac{dJ}{dr} \quad &= nr^{n-1}u'(r)v'(r) + r^n u''(r)v'(r) + r^n u'(r)v''(r) \\
&+(n-1)(n-2)r^{n-2}\left[\alpha u(r)v'(r) + (1-\alpha)u'(r)v(r)\right] \\
&+(n-2)r^{n-1}\left[\alpha u(r)v''(r) + u'(r)v'(r) + (1-\alpha)u''(r)v(r)\right] \\
&+n\beta r^{n-1}u^k(r)v^{p+1}(r) + k\beta r^n u^{k-1}(r)v^{p+1}(r)u'(r) + (p+1)\beta r^n u^k(r)v^p(r)v'(r) \\
&= \left[\beta(p+1) - a\right]r^n u^k(r)v^p(r)v'(r) + \left(\beta k - b\right)r^n u^{k-1}(r)v^{p+1}(r)u'(r) \\
&+ \left[n\beta - a(n-2) - \alpha(n-2)(b-a)\right]r^{n-1}u^k(r)v^{p+1}(r).
\end{aligned}$$

An integration yields the desired result.

Proposition 1 *Suppose that at least one of the solutions $u(r)$ and $v(r)$ of (21) - (23) has a zero. If $a > b$, then $u(r)$ reaches zero first (i.e. there is some R such that $u(R) = 0, v(R) > 0$ and $u(r), v(r) > 0$ for $r < R$). If $b > a$, then $v(r)$ reaches zero first.*

Proof. If $a < b$, we will prove that $v(r)$ reaches zero first. Suppose otherwise $u(R) = 0$ first. Using Lemma 4, we have for $\alpha > 1$

$$0 < R^n u'(R)v'(R) + (n-2)(1-\alpha)R^{n-1}u'(R)v(R) \tag{28}$$

$$= [n\beta - a(n-2) - \alpha(n-2)(b-a)] \int_0^R u^k(s)v^{p+1}(s)s^{n-1}ds$$

$$+ [\beta(p+1) - a] \int_0^R u^k(s)v^p(s)v'(s)s^n ds + (\beta k - b) \int_0^R u^{k-1}(s)v^{p+1}(s)u'(s)s^n ds.$$

Since $v'(s), u'(s) < 0$ in $(0, R)$ by (11), the last two integrals are negative for β sufficiently large. Since $a < b$, we can select α much greater than 1 such that the first term on the right side is very large and negative. Thus, this will lead to a contradiction since the left side is positive. This completes the proof in this case. If $a > b$, we can conclude by applying a similar argument that $u(r)$ reaches zero first.

As a corollary of this proposition, we derive immediately a very interesting nonexistence result for local solutions to the problem (21)-(23).

Theorem 3 *If $a \neq b$, then problem (21) - (23) with Dirichlet boundary condition $u(R) = v(R) = 0$ for any $R > 0$ has no positive radial solutions.*

We now obtain an existence result for the positive radial solutions.

Theorem 4 *Suppose that $k(n-2) \geq n$ and $b(n-2)(p+1) \geq na$ if $a \geq b$ and that $p(n-2) \geq 2$ and $a(n-2)k \geq nb$ if $b > a$. Then, for any positive initial values $u(0)$ and $v(0)$, the solutions $u(r)$ and $v(r)$ of (21) - (23) are positive in $(0, \infty)$.*

Proof. Suppose that the conclusions were not true, then for some $R > 0$, we would have $u(R) = 0$ if $a \geq b$ or $v(R) = 0$ if $b > a$ and $u(r), v(r) > 0$ for $r < R$. First assume $u(R) = 0$. By Lemma 4 with $\alpha = 1$,

$$0 < R^n u'(R)v'(R) = [n\beta - b(n-2)] \int_0^R u^k(s) v^{p+1}(s) s^{n-1} ds$$

$$+ [\beta(p+1) - a] \int_0^R u^k(s) v^p(s) v'(s) s^n ds + (\beta k - b) \int_0^R u^{k-1}(s) v^{p+1}(s) u'(s) s^n ds. \tag{29}$$

Selecting $\beta = b(n-2)/n$, we get a contradiction, since the right side is nonpositive under our assumptions. If $v(R) = 0$, using (27) with $\alpha = 0$ and choosing $\beta = a(n-2)/n$, we also get a contradiction. This completes the proof.

Theorem 5 *If $(n-2)k < n$ and $(n-2)p < 2$, then there are no positive solutions in $(0, \infty)$ for problem (21) - (23).*

Proof. Suppose otherwise that $u(r)$ and $v(r)$ are positive solutions for all $r > 0$. Then $u'(r) < 0$ and $v'(r) < 0$ for $r > 0$. By Lemma 2,

$$\begin{aligned} & r^n u'(r) v'(r) + (n-2) r^{n-1} \left[\alpha u(r) v'(r) + (1-\alpha) u'(r) v(r)\right] \\ = \; & \alpha r^{n-1} v'(r) \left[r u'(r) + (n-2) u(r)\right] + (1-\alpha) r^{n-1} u'(r) \left[r v'(r) + (n-2) v(r)\right] \\ \leq \; & 0, \end{aligned} \tag{30}$$

for $0 \leq \alpha \leq 1$ and all $r > 0$.

First we prove that for any $\varepsilon > 0$, there exists an r_1, such that $J(r) < \varepsilon$ for $r \geq r_1$, where $J(r)$ is defined by (26). In fact, if $k - p - 1 \geq 0$, using (12), we get

$$\begin{aligned} r^n u^k(r) v^{p+1}(r) = \; & r^{n - \frac{2k}{k-1}} [v(r)]^{p+1-\frac{kp}{k-1}} \left[r^2 u^{k-1}(r) v^p(r)\right]^{\frac{k}{k-1}} \\ \leq \; & c\, r^{-\frac{n-(n-2)k}{k-1}} [v(r)]^{\frac{k-p-1}{k-1}} \to 0 \quad \text{as} \quad r \to \infty. \end{aligned} \tag{31}$$

If $k - p - 1 < 0$, we also have

$$\begin{aligned} r^n u^k(r) v^{p+1}(r) = \; & r^{n - \frac{2(p+1)}{p}} [u(r)]^{k - \frac{(p+1)(k-1)}{p}} [r^2 u^{k-1}(r) v^p(r)]^{\frac{p+1}{p}} \\ \leq \; & c\, r^{-\frac{2-(n-2)p}{p}} [u(r)]^{\frac{p+1-k}{p}} \to 0 \quad \text{as} \quad r \to \infty. \end{aligned} \tag{32}$$

Combining (30) - (32), we get $J(r) < \varepsilon$ for $r \geq r_1$.

Next, we prove that the right side of (27) is greater than a positive constant. In fact, if $b(p+1) \leq ak$, applying Lemma 4 with $\beta = b/k$ and $\alpha = 1$ yields

$$J(r) = \frac{b[n-(n-2)k]}{k} \int_0^r u^k(s) v^{p+1}(s) s^{n-1} ds$$
$$+ \frac{b(p+1)-ak}{k} \int_0^r u^k(s) v^p(s) v'(s) s^n ds \geq c_1 > 0.$$

If $b(p+1) > ak$, applying Lemma 4 with $\beta = a/(p+1)$ and $\alpha = 0$ also yields

$$\begin{aligned} J(r) \quad &= \frac{a[2-(n-2)p]}{p+1} \int_0^r u^k(s) v^{p+1}(s) s^{n-1} ds \\ &+ \frac{ak - b(p+1)}{p+1} \int_0^r u^{k-1}(s) v^{p+1}(s) u'(s) s^n ds \geq c_0 > 0. \end{aligned}$$

Since $J(r) < \varepsilon$ for $r \geq r_1$, we get a contradiction. Hence, the theorem follows.

We now apply lemmas 1 and 3 to prove the following

Theorem 6 *Let $l = n+2-(n-2)(k+p) > 0$. Suppose that $k-1 > p$ if $b \geq a$ and that $k-1 < p$ if $a \geq b$. Then, there are no positive solutions in $(0, \infty)$ for problem (21) - (23).*

Proof. We only give the proof when $b \geq a$. Suppose that the conclusions are not true, and $u(r),\ v(r)$ are the positive solutions of (21) - (23) in $(0, \infty$. From Lemma 3, we have $u(r) \geq c_0 v(r)$. By Lemma 1,

$$c_0^{k-1} v^{k-1+p}(r) \leq u^{k-1}(r) v^p(r) \leq c\, r^{-2},$$

which implies that

$$v(r) \leq c\, r^{-\frac{2}{k-1+p}}. \tag{33}$$

Denote

$$M_j = \int_0^j u^k(s) v^{p+1}(s) s^{n-1} ds.$$

Using (24) and (33) we obtain

$$\frac{u(r)}{v(r)} \geq \int_1^r \frac{b-a}{\tau^{n-1} v^2(\tau)} \int_0^1 u^k(s) v^{p+1}(s) s^{n-1} ds d\tau,$$

$$= M_1 \int_1^r \frac{b-a}{\tau^{n-1} v^2(\tau)} d\tau \geq c \int_1^r \tau^{\frac{4}{k-1+p}+1-n} d\tau$$

$$= \frac{c(k-1+p)}{n+2-(n-2)(k+p)} \Big(r^{\frac{n+2-(n-2)(k+p)}{k-1+p}} - 1 \Big) \geq c\ r^{\frac{l}{k-1+p}} \quad \text{for} \quad r \geq 2. \tag{34}$$

Using Lemma 1 and (34), we obtain

$$c_0^{k-1}\ r^{\frac{l(k-1)}{k-1+p}}\ v^{k-1+p}(r) \leq u^{k-1}(r) v^p(r) \leq c\ r^{-2},$$

which implies that

$$v(r) \leq c_1\ r^{-\frac{2}{k-1+p} - \frac{l(k-1)}{(k-1+p)^2}}.$$

Now we use induction to prove that

$$v(r) \leq c_j\ r^{-\mu(j)}, \quad \text{for} \quad r \geq 2^j, \ \ j \geq 1, \tag{35}$$

where

$$\mu(j) = \frac{2}{k-1+p} + \frac{l(k-1)}{(k-1+p)^2} + \cdots + \frac{2^{j-1} l(k-1)^j}{(k-1+p)^{j+1}}.$$

Suppose that (35) holds for $j \geq 1$. By (24),

$$\frac{u(r)}{v(r)} \geq M_{2^j} \int_{2^j}^r \tau^{1-n+2\mu(j)} d\tau = c\ \left(r^{\beta(j)} - 2^{j\beta(j)} \right) \geq c\ r^{\beta(j)}, \quad \text{for} \quad r \geq 2^{j+1}, \tag{36}$$

where

$$\beta(j) = 2 - n + 2\mu(j) = \frac{l}{k-1+p} + \frac{2l(k-1)}{(k-1+p)^2} + \cdots + \frac{2^j l(k-1)^j}{(k-1+p)^{j+1}} > 0.$$

Using (36) and Lemma 1 we obtain

$$c^{k-1} r^{\beta(j)(k-1)} v^{k-1+p}(r) \leq u^{k-1}(r) v^p(r) \leq c\ r^{-2},$$

which implies that

$$v(r) \leq c_{j+1}\ r^{-\frac{2+\beta(j)(k-1)}{k-1+p}} = c_{j+1}\ r^{-\mu(j+1)}.$$

Hence (35) holds for all $j \geq 1$.

Since

$$\begin{aligned}\mu(j) &= \frac{2}{k-1+p} + \frac{l}{2(k-1+p)} \sum_{i=1}^{j} \left(\frac{2(k-1)}{k-1+p}\right)^i \\ &= \frac{2}{k-1+p} + \frac{l(k-1)}{(k-1+p)(k-1-p)} \left[\left(\frac{2(k-1)}{k-1+p}\right)^j - 1\right]\end{aligned}$$

and

$$c\, r^{-(n-2)} \le v(r) \le c_j\, r^{-\mu(j)} \quad \text{for} \quad r \ge 2^{j+1},$$

if j is large enough, this leads to a contradiction. Thus we have completed the proof.

Remark. From the Figure given in the introduction, we notice a gap between existence and nonexistence of positive solutions. Numerical calculations show that, for some combinations of a, b, p and q located in this gap, the initial values can determine whether the corresponding solutions are always positive or not.

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